# Some Applications of Generalized Char-Sets of Ordinary Differential Polynomial Sets

Farkhanda Afzal

*School of Mathematics and Systems Science, Beihang University, Beijing, China*

**Abstract.** The notion of characteristic sets, which are a special kind of triangular sets, is introduced by J. F Ritt and W.T. Wu. Wu extended Ritt's work and developed the characteristic set method not only in theory but in algorithms, efficiency and its numerous applications. Triangular sets are widely considered as a good representation for the solution of polynomial systems. After the introduction of characteristic sets by Ritt, triangular sets have become an alternative tool for representing the ideal besides the Gröbner bases. This paper is about implementation and applications of generalized characteristic sets of ordinary differential polynomial sets defined by author.

## 1 Introduction

The key inclination in the information era is the mechanization of mental labor with the assistance of computers. Partial systematization of mental labor allows scientists and engineers to free themselves from tedious and sometimes human unreachable tasks, to ponder on high-level innovative activities and hence to greatly enhance social productivity. The objective of mathematics mechanization is to cultivate symbolic algorithms for manipulating mathematical objects. It also helps in proving and discovering theorems in a mechanical mode. Applications and multidisciplinary studies are major advantages for attraction of scientists in mathematics mechanization.

J. F. Ritt [11] introduced the concept of a characteristic set of a finite or infinite set of differential polynomials. One of his objective was to provide a method to solve systems of differential equations. Characteristic sets of prime ideals have good properties but Ritt's process involves factorization in field extensions. Wu Wen-tsun used Ritt's work to provide an algorithm for solving systems of algebraic equations by means of triangular sets which only requires pseudo-remainder computations. Therefore Characteristic Set Method of Wu has released Ritt's decomposition from polynomial factorization, opening access to a variety of discoveries in polynomial system solving. This method and the methods for geometry reasoning and computation have various applications.

The work of Wu has been extended towards more powerful decomposition algorithms and applied to different types of polynomial systems or decompositions. In addition to be a powerful tool for Wu's general theory and method of mechanical theorem proving, the

characteristic set method has proved efficient for solving a wide class of problems in geometry and algebra.

Dongming Wang [4] proposed an effective strategy to improve Ritt-Wu's algorithm of characteristic sets and a comprehensive implementation of Ritt-Wu's method in the Maple system was described by him. Wang widespread Ritt-Wu's algorithm by means of one-step pseudo-reduction with strategies for the selection of reductends and optimal reductors. In addition to Ritt-Wu's methods, there are many other proficient methods for decomposing systems of multivariate polynomials. Kalkbrener [13] familiarized the notion of regular chain and given a way for decomposing any algebraic variety into unmixed-dimensional components represented by regular chains.

Meng et al [12] follow the work of Wang to present a new algorithmic scheme for computing generalized characteristic sets competently. By familiarizing admissible reductions other than pseudo-division, this is possible to control the swell of coefficients of intermediate polynomials and to compute with smaller polynomials.

We have computed the generalized characteristic sets of Wang for the ordinary differential polynomial sets [7]. Let C be a d-char set of d-polynomial P. A necessary condition is that all the d-polynomials in the set P have d-pseudo remainder zero 0 w.r.t the set C. To make the concept of d- char generalized, this necessary condition is weakened by the replacement of d-polynomial P with an arbitrary d-polynomial set which can generate the same ideal as the d-polynomial set. In [8] we gave a comparison of d-char sets with generalized d-char sets and results of preliminary experiments show that generalized d-char sets perform better than d-char sets

based on d-pseudo division in efficiency and simplicity of outcome.

# 2 Preliminaries

We can classify triangular decompositions algorithms for polynomial systems into several ways. One may first consider the relation between the input polynomial system and the output triangular system. In that perspective, we have two types of triangular decompositions that are basically different. Those systems for which output triangular system can encode all the points of the zero set of input polynomial system. The second are those for which triangular system represents only the generic zeros of the irreducible components of the input polynomial system. Similarly, triangular decomposition algorithms can be classified with the help of algorithmic principles on which they are relying. From this angle we have two different types of triangular decompositions. One is followed by variable elimination i.e., reduce the solving of a polynomial system in $n$ unknowns to a system of $n - 1$ unknowns. Second is the type that proceed inclemently i.e., reduce the solving of a polynomial system in $m$ equations to that of a polynomial system in $m - 1$ equations. The characteristic set method fits in to first type in each classification describe above. Mostly, works on triangular decomposition algorithms focus on incremental solving as this principle is relatively attractive, because it allows controlling the properties and growth of the intermediary computed objects.

## 2.1 Generalized differential characteristic sets

The char-set method of polynomial equations-solving is naturally extended to the differential case which gives rise to an algorithmic method of solving arbitrary systems of differential equations.

We use notion of the differential medial set (d-medial set) of the d-polynomial set $\mathbb{P}$. Let $\mathbb{P}$ be a non-empty d-polynomial set $\mathcal{K}\{x\}$.

Definition 2.1:A d- ascending set $\mathbb{M}$ is called a d-medial set of $\mathbb{P}$, if $\mathbb{M} \subset \langle \mathbb{P} \rangle$ and $\mathbb{M}$ has ranking not higher than the ranking of any d-basic set of $\mathbb{P}$.

The d-medial sets are the d-ascending sets with rank not higher than that of the d-basic set of $\mathbb{P}$ and in which all the d- polynomials are linear combinations of the d-polynomials in $\mathbb{P}$ with d-polynomial coefficients. Therefore any d-basic set itself is a special d-medial set of the d-polynomial set. It has been proved that in Ritt's original algorithm, the d-basic set can be replaced by the d-medial set.

For any non-empty d-polynomial set $\mathbb{P}$ in $\mathcal{K}\{x\}$, an ascending set $\mathbb{C}$ is called a generalized differential characteristic (generalized d-char) set if it satisfied the following two conditions
(1) $\mathbb{C} \subset \langle \mathbb{P} \rangle$
(2) There exists a d-polynomial set $\mathbb{Q} \subseteq \mathcal{K}\{x\}$ such that $\langle \mathbb{Q} \rangle = \langle \mathbb{P} \rangle$ and d-prem $(\mathbb{Q}, \mathbb{C}) = 0$

If we recall, for C to be a d-char set of a d-polynomial set $P$, a necessary condition is that all the d-polynomials in $P$ have pseudo-remainder 0 w.r.t. $C$.

This compulsory condition is weakened by replacing the given d-polynomial set with an arbitrary d-polynomial set which generates the same ideal as the input d-polynomial set, leading to the concept of generalized characteristic set [12]. This set may have d- polynomials of degrees smaller than the degrees of those in the input set and consequently may take less computing time for pseudo-reduction to 0. Therefore, an algorithmic scheme for computation of generalized characteristic sets is presented in [7].

This scheme is defined by several admissible d-reductions. For this purpose following admissible reductions have been introduced [8].

### 2.1.1 Univariate GCD d-reduction.

Univariate GCD d-reduction is defined as follows d-Rem $(P, Q, \mathfrak{D}_{UG}) :=$

$$:= \begin{cases} [0, \gcd(P, Q, x_q)] & (i) \\ [P, Q] & (ii) \end{cases}$$

(i)      If $P, Q$ are univariate polynomials in $x_q$,

(ii)     otherwise ,

where $x_q$ is some variable in x.

### 2.1.2 One-step d-pseudo-division reduction.

One-step d-pseudo-division reduction is defined as follows:

$$\text{d-Rem}(P, Q, \mathfrak{D}_{SP}) := \begin{cases} [d - \text{stprem}(P, Q), Q] \\ [P, Q] \end{cases}$$

if P be d-reducible w.r.t Q; then
   d-stprem $(P, Q) \in \langle P, Q \rangle$.
Therefore
   d-Rem$(P, Q, \mathfrak{D}_{SP}) \subseteq \langle P, Q \rangle$.
Otherwise
   d-Rem$(P, Q, \mathfrak{D}_{SP}) = [P, Q] \subseteq \langle P, Q \rangle$ is obvious.

### 2.1.3 One-step d-division reduction.

The division operation can also be viewed as an admissible reduction. So we define the one-step d-division reduction.

$$\text{d-Rem}(P, Q, \mathfrak{D}_{SD}) := P \begin{cases} \left[ P - \frac{M}{\text{lead}(Q)} \cdot Q, Q \right] & (i) \\ \left[ P - \frac{M}{\text{lead}(Q')} \cdot Q', Q \right] & (ii) \\ [P, Q], & (iii) \end{cases}$$

(i)     if there exists a monomial M of P and such that $\text{lead}(Q) | M$,

(ii)    if there exists a monomial M of P, such that $\text{lead}(Q') | M$,

(iii)   otherwise.

### 2.1.4 Subresultant d-PRS reduction.

We introduce the following d-reduction for d-polynomials.

$$\text{d-Rem}(P, Q, \mathfrak{D}_{PD}) := \begin{cases} [0, P_r] & (i) \\ [P_r, P_{r-1}] & (ii) \\ [P, Q] & (iii) \end{cases}$$

(iv)    if lead $(P) = \text{lead}(Q), \text{ld}(P) = \text{ld}(Q)$, and d-res $(P, Q, \text{lead}(Q)) = 0$,

(v)    if lead $(P) = \text{lead}(Q), \text{ld}(P) = \text{ld}(Q)$, and d-res $(P, Q, \text{lead}(Q)) \neq 0$,

(vi)    otherwise.

With the help of admissible d-reductions above, an algorithmic scheme NewdCharSet for computing generalized d-char sets for ordinary d-polynomial sets is introduced with two sub sequent algorithms dMedSet and dFind3R (Reader may refer to [7] for complete details).

## 2.2 Applications of generalized differential characteristic sets

The previous sections introduced generalized d-char sets without their applications taking into consideration. However, this approach permitted to introduce the majority of the necessary formalisms. In this section we describe the purpose of d-char sets, which is to attain information about sets or systems of differential equations d-charset methods are applied to set or systems of differential equations in order to obtain information about them. What type of information is gained does not only depend on the differential polynomial equations, but also on the used differential ranking on the indeterminates. in general, d-char set computations are used to decouple indeterminates, eradicate high orders, and segregate dependencies. However, d-char sets do not yield solutions to set or systems of differential polynomial equations. Their computations lead to simpler sets or systems which are in general easier to solve. Thereby, the computations assist in solving set and systems of differential equations but do not solve the set or systems directly. Just as Grbner bases help in solving systems of algebraic equations and do not resolve the systems directly. D-char set computations transform a set system of differential equations into another set or system. Obviously, the solutions should not change.

*Example 3.1*: [2]. Use the same coordinate system and differential equations for Kepler's law. Suppose we have same $H_1, H_2, H_3$ as

$$H1 = r2 - x2 - y2 = 0,$$
$$H2 = a2 - x'\,'\,2 - y'\,'\,2 = 0,$$
$$H3 = x'\,'\,y - y'\,'\,x = 0 \text{ and}$$
$$K = x''r' - r''x'.$$

Let we have d-polynomial set
$\mathbb{E} = \{H_1, H_2, H_3, K\}$ with $r \prec a \prec x \prec y$.

By computing a d-char set of a d-polynomial set $\mathbb{E}$ by the, one obtain the following d-char set:
$\{r^2x^2(r'\,'\,^4r^2 - 2r'\,'\,^2r^2a^2 + r^2a^4 + r'\,'\,^4x^2 - r'\,'\,^2x^2a^2), x(ry'\,' - r'\,'\,y), x^2r^2r'\,'\,^2(r^2a^2 + r'\,'\,^2y^2 - y^2a^2 - r^2r'\,'\,^2), r^2x^2(-r'\,'\,^2 + a^2)\}$

*Example 3.2*: (Bertrand curves) A pair of space curves having their principal normals in common are said to be associate Bertrand curves. Let $C$ and $\bar{C}$ be Bertrand pair of curves in one-to-one correspondence with arc lengths s, S as parameters in the ordinary metric space. Attach the trihedral $(X, e_1, e_2, e_3)$ to $C$ at point $X$ and trihedral $(\bar{X}, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ to $\bar{C}$ at corresponding point $\bar{X}$. Denote the

curvature and torsion of $C$ and $\bar{C}$ by $\kappa, \tau$ and by $\bar{\kappa}, \bar{\tau}$ respectively. We have the following theorems.

**Schell's Theorem**. The product of $\tau$ and $\bar{\tau}$ is a constant.

**Betrand's Theorem**. There exist a linear relation between $\kappa$ and $\tau$ with constant coefficients.

**Mannheim's Theorem.** The cross-ratio of $X, \bar{X}$ and the centers of $\kappa, \bar{\kappa}$ is a constant.

Using the orthogonality relations (see details [23]), one obtain a set of fourteen d-polynomials.

$$U_1 = \bar{s}\,u_{11} + a_2\,\kappa - 1, \quad U_2 = -a'_2,$$
$$U_3 = \bar{s}\,u_{13} + a_2\,\tau, \qquad U_4 = -u'_{11}$$
$$U_5 = \bar{s}\bar{\kappa} - \kappa\,u_{12} + \tau\,u_{13}, \quad U_6 = -u'_{13}$$
$$U_7 = \bar{s}\bar{\kappa}\,u_{11} - \bar{s}\bar{\kappa}, u_{31} - \kappa, \quad U_8 = \bar{s}\bar{\kappa}\,u_{13} - \bar{s}\bar{\kappa}, u_{33} + \tau,$$
$$U_9 = -u'_{31}, \qquad U_{10} = -\bar{s} - \kappa\,u_{31} + \tau\,u_{33},$$
$$U_{11} = u'_{31}, \qquad U_{12} = u_{11} - u_{33},$$
$$U_{13} = u_{13} + u_{31}, \qquad U_{14} = u_{11}^2 + u_{13}^2 - 1$$

With the order
$$\bar{s} \prec a_1 \prec a_2 \prec a_3 \prec u_{11} \prec u_{12} \prec u_{13} \prec u_{21} \prec u_{22} \prec u_{23}$$
$$\prec u_{31} \prec u_{32} \prec u_{33} \prec \kappa \prec \tau \prec \bar{\kappa} \prec \bar{\tau}$$

Let
$$\mathbb{U} = \{U_1, U_2, \ldots, U_{14}\}$$
be a d-polynomial set. If we compute a d-char set of a d-polynomial set $\mathbb{U}$ then the output consists of the following d-char set:
$\{-a'\,^2, s'\,'\,s'\,u^{11}(u_{11}^2 - 1), s'\,'\,u_{11}(-s'\,-u_{12} + u_{12}s'\,u_{11} + u_{11}), s'\,'\,u_{11}\,u_{13}(-u_{11} + s'\,), u_{31}\,u_{11}\,s'\,'\,(-u_{11} + s'\,), u_{11} - u_{33}, s'\,u_{11} + a_2\,\kappa - 1, s'\,'\,a_2\,\tau\,u_{11}(-u_{11} + s'\,), s'\,'\,u_{11}(a_2\,s'\,\bar{\kappa} + s'\,-u_{11}), s'\,'\,a_2\,s'\,\bar{\tau}\,u_{11}(-u_{11} + s'\,)\}$.

*Example 3.3*: The y-axis projection of the velocity of a particle moving within a plane is a constant c. If p is the curvature radius of the orbit of the particle. By a formula for curvature in differential geometry, we have:

$$V_1 = a - x'\,' = 0,$$
$$V_2 = v^2 - x'\,'\,^2 - c^2 = 0,$$
$$V_3 = p^2\,c^2\,x'\,'\,^2 - (y'\,'\,^2 + x'\,'\,^2)^3 = 0.$$

Consider a d-polynomial set $\mathbb{V} = \{V_1, V_2, V_3\}$. Fix an order $c \prec x \prec v \prec a \prec p$ with $c$ as constant parameter. Compute a d-char set of the set $\mathbb{V}$ by the algorithm NewdCharSet. We get following d-polynomial set as the output:
$\{a - x'\,', -v^2 + x'\,'\,^2 + c_2, p^2c^2x'\,'\,^2 - y'\,^6 - 3\,y'\,^4x'\,^2 - 3\,y'\,^2x'\,^4 - x'^6\}$.

*Example 3.4:* Consider the d-polynomial set $W$ which consists of the following four d-polynomials:

$$W_1 = c_1\,x'\,'\,^2 + c_2\,x'\,' + c_3,$$
$$W_2 = y^2 - (e^2 - 1)\,x^2 - 2\,p\,e\,x - p^2,$$
$$W_3 = r - p - e\,x,$$
$$W_4 = r^2 - h.$$

where $c_1$, $c_2$ and $c_3$ are polynomials of $p$, $e$, $h$, $x$ and $x'$. Consider the following order $p \prec e \prec h \prec x \prec y \prec r \prec r$ with $p$, $e$ and $h$ as constant. After computing a d-char set of a d-polynomial set $\mathbb{W}$ by the NewdCharSet, one gets the following d-char set in the output:
$\{-h + ap^2 + 2\,paex + e^2x^2a, -r + p + ex, c_1\,x'\,'\,^2 + c_2\,x'\,' + c_3, -y^2 + x^2e^2 - x^2 + 2\,pex + p^2\}$.

*Example 3.5:* The equations describing a motion under central force can be formulated as below:

$$x'' y - y'' x = 0,$$
$$r^2 - x^2 - y^2 = 0,$$
$$h - x^2 y' + x' y = 0,$$
$$r^2 w' - h = 0.$$

where $h$ and $w$ stands for the angular momentum per unit mass and the inclination of the radius to the $x$-axis respectively.

If we take the following order $x \prec\prec y \prec r \prec h \prec w \prec v \prec a$ and consider

$\mathbb{X} = \{x'' y - y'' x, r^2 - x^2 - y^2, h - x^2 y' + x' y, r^2 w' - h\}$.

Then a d-char set of the d-polynomial set $\mathbb{X}$, is given as follows:

$\{w' x^2 + w' y^2 - xy' + x' y, -x'' y + y'' x, h - xy' + x' y, r^2 - x^2 - y^2\}$

The examples described above have been solved by using epsilon package in Maple 14 on a laptop having windows 7 ultimate, Intel(R)Core(TM)2 Duo CPU 2.4 GHz with 2.00 GB memory.

Besides the determination of differential solvability and differential radical ideal membership, we list a number of applications related to differential characteristic sets.

The differential characteristic set method generalized a number of well-known algorithms in symbolic computations

They can be used to compute the differential dimension of a differential algebraic set.

The capabilities of the non linear differential sets or system solver, dsolve, that is improved in maple 6 by using the diffalg package. In order to solve a single differential equation, the characteristic decomposition demonstrates the differential equations for their singular solution. If they are resolved, the complete solution set is currently returned by dsolve. The straightforward design for solving ordinary differential sets or systems is to attain advantage of the solver for single non linear differential equation. The contributed sets or system is therefore decomposed with respect to an elimination ranking. Afterward to solve the sets or system in close form by solving iteratively single differential equations in a distinctive indeterminate.

One more application of differential characteristic decomposition algorithms in the direction of solving ordinary differential equations is the suggestion is to generate new classes of ordinary differential equations, for example of first order, which may be solved jointly with a way of distinguishing whether a specified ordinary differential equation is in the class or not.

The variants of the differential characteristic decomposition algorithms are used to examine the several biological models that are arising in the literature.

The work of several mathematician showed the relevance of constructive differential algebra in control theory.

A foundation of over-determined sets or systems of partial differential equations comes up to symmetry analysis. The seminal equations of the insignificant generators of the Lie symmetry group of a differential equation form a set of linear partial differential equations. Reducing this set permits to establish if we have symmetry, to investigate the configuration of the group and apply that knowledge to classification troubles or in order to discover a reduction of the original equation. Since the system is linear, there are a number of substitute algorithms that may be applied.

## 3 Tables

The Table I provides information about the test examples. We have number of terms (nops) in each d-polynomial which appears in d-char set of each example, the time in seconds for computing d-char set for each example. Table II provides information about degree tuple of each d-polynomial obtained in the output of d-char set

**Table 1.** nops and timing in the outputs

| No | nops | time |
|----|------|------|
| 1 | 5, 2, 4, 2 | 0.062s |
| 2 | 1, 2, 4, 2, 2, 2, 3, 2, 3, 2 | 0.187s |
| 3 | 3, 2, 5 | 0.016s |
| 4 | 4, 3, 3, 5 | 0.031s |
| 5 | 4, 2, 3, 3 | 0.032s |

**Table 2.** deg tuple of d-polynomial set in the output of NewdCharSet

| No | degree tuple |
|----|--------------|
| 1 | 4, 4, 4, 0],[1, 0, 1, 1],[4, 2, 2, 2],[2, 2, 2, 0] |
| 2 | [0,..., 0], [0, 0, 0, 3, 0,..., 0],[0, 0, 0, 0, 2, 1, 0,..., 0] <br> [0, 0, 0, 0, 2, 0, 1, 0,..., 0], [0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 1, 0,..., 0] <br> [0, 0, 0, 0, 1, 0,..., 0, 1, 0, 0, 0, 0],[0, 0, 1, 0, 1, 0,..., 0, 1, 0, 0, 0] <br> [0, 0, 1, 0, 2, 0,..., 0, 1, 0, 0], [0, 0, 1, 0, 2, 0,..., 0, 1, 0], <br> [0, 0, 1, 0, 2, 0,..., 0, 1] |
| 3 | [0, 0, 0, 1, 0],[2, 0, 2, 0, 0], [2, 0, 0, 0, 2] |
| 4 | [2, 2, 1, 2, 0, 0, 1],[1, 1, 0, 1, 0, 1, 0], <br> [0, 0, 0, 0, 0, 0, 0], <br> [2, 2, 0, 2, 2, 0, 0] |
| 5 | [2, 2, 0, 0, 0, 0, 0], [1, 1, 0, 0, 0, 0, 0], <br> [1, 1, 0, 1, 0, 0, 0] <br> [2, 2, 2, 0, 0, 0, 0] |

## 4 Conclusion and future work

We conclude the paper with the the discussion that computation of generalized differential characteristic set leads to simpler sets or systems of differential equations for which find solution is much easier as compare to the original set or systems.

A differential characteristic set of differential ideal is a finite subset of d-polynomial sets from which many

properties are often obtain by the inspection of its elements. Computation of differential characteristic sets is constructive in principle.

We have generalized the differential characteristic sets for ordinary differential case. This concept will be further improved and compared with more schemes by different strategies in the forthcoming papers. To extend the idea of generalized differential characteristic sets to partial differential case in also in our considerations. By a numerical superscript corresponding to the same superscript after the name of the author concerned. Please ensure that affiliations are as full and complete as possible and include the country.

## Acknowledgment

## References

[1] S.C. Chou, X.S. Gao. Automated reasoning in mechanics using Ritt-Wu's method. University of Texas at Austin, USA. (1989).

[2] S.C. Chou, X.S. Gao Automated reasoning in mechanics using Ritt-Wu's method; PartIII. Proceedings of the IFIP International Work-shop on Automated Reasoning Elsevier Science Publishers Beijing 1-12. (1992)

[3] S.C. Chou, X.S. Gao. Automated Reasoning in Differential Geom- etry and Mechanics using characteristic method IV Bertrand curves. J. of Sys. Math 6:186-192. 216–228. (1993)

[4] D. Wang. A generalized algorithm for computing characteristic sets,World Scientific Publishing Company, Singapore, 165–174. (2001)

[5] D. Wang. Elimination methods, texts and monograms in symbolic com- putations, Springer-Verlag, Wien New York. (2001)

[6] E. Hubert. Notes on triangular sets and triangulation-decomposition algorithms II: diferential systems, Lecture Notes in Computer Science, Springer-Verlag, Berlin, 2630, 40–87. (2003)

[7] F. Afzal. Generalized char sets for ordinary differential polynomial sets, Science International, **28 (3)**, (2016).

[8] F. Afzal. Comparison of Char-Sets with Generalized Char-Sets of Ordinary differential Polynomial Sets, International Conference on Applied Computing, Mathematical Sciences and Engineering (ACME), Johor Bahru, Malaysia. (2016)

[9] Gallo, G., Mishra, B, Efficient algorithms and bounds for Wu-Ritt characteristic sets, In: Mora, T., Traverso, C. (Eds.), Efective Methods in Algebraic Geometry, Progress in Mathematics, Birkhauser, Berlin, **94**, 119–142. (1991)

[10] G. Gallo, B. Mishra, Wu-Ritt. characteristic sets and their complexity. Discrete Mathematics and Theoretical Computer Science, American Mathematical Society, Providence, **6**, 1110–136. (1999)

[11] J. F. Ritt. Differential algebra, New York: AMS Press. (1950)

[12] J. Meng, L. Xiaoliang, D. Wang. A new algorithmic scheme for com- puting characteristic sets, Journal of Symbolic Computation, **50**, 431–449. (2013)

[13] Kalkbrener, M. A, generalized Euclidean algorithm for computing triangular representations of algebraic varieties, Journal of Symbolic Computation, **15**, 14–167. (1993)

[14] M. Fliess, Generized controller canonical forms for linear and nonlinear dynamocs, IEEE Trans.Automat. Control **35**, 994-1001. (1990)

[15] Li, Y, Some properties of triangular sets and improvement upon al- gorithm CharSer, In: Calmet, J., Ida, T., Wang, D. (Eds.), Artificial Intelligence and Symbolic Computation, Lecture Notes in Computer Science. Springer-Verlag, Beijing, **4120**, 82–93. (2006)

[16] S. Diop and M. Fliess, Nonlinear observability, identifitability and persistent trajesctories, in: Proc. 30th IEEE confernce on Decision and control(IEEE Press), New York, 714-719. (1991)

[17] T. Sasaki, A. Furukawa. Theory of multiple polynomial remainder sequence, Publ. RIMS, Kyoto Univ, **20**, 367–399. (1984)

[18] W. T. Wu. Mathematics-mechanization research preprints, No 1-6, MM Research Center, Academia Sinica, **986**. (1987-1991)

[19] X.S. Gao, J. Van der Hoeven, Y. Luo, and C. Yuan. Characteristic set method for differential-difference polynomial systems. Journal of Symbolic Computation, **44**, 1137–1163. (2009)

[20] X.S. Gao and Z. Huang, Characteristic set algorithms for equation solving in finite fields and applications in cryptanalysis, Journal of Symbolic Computation, **47**, 655–679. (2012)

[21] Y. Chen, X. S. Gao. Involutive characteristic sets of algebraic partial diferential equation systems, Science in China Series A: Mathematics, **46**, 469–487. (2003)

[22] Z. W. Gan, M. Zhou,Decomposition of Reflexive Differential-Difference Polynomial Systems, Applied Mechanics and Materials, Vols 380-384, 1645-1648. (2013)