

# STUDYING PERIODIC ROLLING WAVES ON THE SURFACE OF A FALLING LIQUID FILM

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**Abstract.** In this paper the mathematical model describing the falling thin liquid film on a vertical wall is considered taking into account heat and mass transfer at the interface in the regime of periodic rolling waves. The families of exact and numerical generalized solutions, where periodic traveling waves conjugate through the strong and weak discontinuities with each other or with the residual thickness, are constructed. Evolution of exact periodic generalized solutions is studied in time.

## 1 Problem statement

Heat transfer at condensation of immobile saturated vapor on a vertical surface was firstly considered in [1] for the case of the laminar flow of a condensate film. Later, theoretical and experimental studies of the film flows, including those with consideration of heat and mass transfer on the free surface, were carried out in many papers. It is shown [2] that the Kapitza waves are not the capillary, but the rolling ones. A possibility of traveling wave existence on the surface of a vertically falling liquid film without consideration of surface tension is shown in [3], and the equation, describing propagation of these waves, so-called kinematic equation, is also derived there (exact discontinuous solutions to this equation correspond qualitatively to the experimental results). In [4], the flow of a thin liquid film on a vertical wall was studied theoretically with consideration of condensation at the interface in the regime of the rolling waves. The families of discontinuous solutions were derived, where the traveling waves conjugate with each other or with a residual thickness through strong and weak discontinuities. In [5] the model that takes into account heat and mass transfer at the interface of a thin film of liquid flowing down a vertical wall in the regime of rolling waves for both cases – evaporation and condensation was studied. The full families of exact generalized solutions, which model the increasing and decreasing waves as well as the rolling waves, where the traveling waves are interfaced through strong or weak discontinuities with each other or with the “residual” thickness were constructed. Time evolution of these families of exact generalized solutions was studied. The maps of flow regimes of the liquid film on a vertical heat transfer surface were plotted. Based on the model developed in [3-5], the traveling waves on the surface of a falling liquid film were studied in [6], taking into account heat and mass transfer and surface tension.

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In this paper, we consider the model that takes into account heat and mass transfer at the interface of a thin film of liquid flowing down a vertical wall in the regime of periodic rolling waves for both cases – evaporation and condensation. We have constructed the exact generalized solutions, which model the increasing and decreasing waves as well as the periodic rolling waves, where the traveling waves are conjugated through strong or weak discontinuities with each other or with the “residual” thickness.

The rolling wave equation was obtained in [5]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left[ \frac{h^3}{3} \left( 1 + \alpha \frac{\partial^3 h}{\partial x^3} \right) \right] = \frac{\beta}{h}, \tag{1}$$

where  $h$  is the liquid film thickness, and the coefficients are determined in the following manner:

$$\alpha = \frac{\sigma h_0}{g \rho l^3} \quad \beta = \frac{(T_s - T_w) \nu \lambda l}{\rho g r h_0^4}.$$

This is the rolling wave equation, which considers the capillary forces ( $\alpha > 0$ ) and condensation at  $\beta > 0$ , when  $T_s > T_w$ , or evaporation at  $\beta < 0$ , when  $T_s < T_w$ . Let us estimate these parameters for water at  $T_s = 310$  K,  $T_w = 300$  K,  $l = 2 \cdot 10^{-2}$  m,  $h_0 = (\nu^2 / g)^{1/3} = 0.5 \cdot 10^{-4}$  m,  $g = 9.8 \text{ m/s}^2$ ,  $\nu = 10^{-6} \text{ m}^2/\text{s}$ ,  $\rho = 10^3 \text{ kg/m}^3$ ,  $\sigma = 72 \cdot 10^{-3} \text{ N/m}$ ,  $\lambda = 0.6 \text{ W/(mK)}$ ,  $r = 2.4 \cdot 10^6 \text{ J/kg}$ . Calculation gives the values  $\alpha = 4.6 \cdot 10^{-5}$ ,  $\beta = 1$ . The discontinuous solutions to equation (1), limit at  $\alpha \rightarrow 0$ , are the stable generalized solutions to hyperbolic equation

$$\frac{\partial h}{\partial t} + \frac{1}{3} \frac{\partial h^3}{\partial x} = \frac{\beta}{h}. \tag{2}$$

The similar equation was derived in [5].

## 2 Periodic solution of equation of the rolling wave at phase transition

The hyperbolic equation (2) admits exact solution  $h = H(t, \eta, \beta) = \sqrt{2\beta t + \eta^2}$ , which describes a change in initial film thickness  $H(0, \eta) = \eta$  in time. At  $\beta > 0$ , i.e., at condensation, the film thickness increases, and at  $\beta < 0$ , i.e., at evaporation, the film thickness decreases. We will derive the exact solutions, which model evolution of a single rolling wave at  $\beta < 0$ , i.e., at evaporation. These solutions are set by formula

$$h = \begin{cases} \tilde{H}(t, \eta_l, \beta), & -\infty < x \leq a(t), \\ h_-(x, t, x_0^-, V_-, \beta), & a(t) \leq x < b(t), \\ h_+(x, t, x_0^+, V_+, \beta), & b(t) \leq x \leq c(t), \\ \tilde{H}(t, \eta_r, \beta), & c(t) \leq x < +\infty \end{cases}. \tag{3}$$

and they are presented [5] by increasing  $h_-$  and decreasing  $h_+$  waves, traveling with velocities  $V_-$  and  $V_+$ , conjugated with depths  $\tilde{H}(t, \eta_l, \beta)$ ,  $\tilde{H}(t, \eta_r, \beta)$  through weak discontinuities on lines  $x = a(t)$  and  $x = c(t)$ , where

$$\tilde{H}(t, \eta, \beta) = \begin{cases} H(t, \eta, \beta), & 0 \leq t \leq T(\eta, \beta), \\ 0, & t \geq T(\eta, \beta). \end{cases} \quad (4)$$

Here,  $T(\eta, \beta) = \eta^2 / (2|\beta|)$  is time, required for complete evaporation of the residual layer with initial depth  $\eta$ . It follows from formulas (3)–(4) that complete evaporation of the residual layer from the left of the rolling wave, i.e., at  $x \leq a(t)$ , occurs at  $T(\eta_l, \beta)$ , and from the right of this wave, i.e., at  $x \geq c(t)$ , it occurs at  $T(\eta_r, \beta)$ .

We will study time evolution of infinite succession of waves of type (3) rolling in the residual layer  $H(t, \eta, \beta)$ , where  $\eta_l = \eta_r = \eta$ , in the regime of evaporation at  $\beta < 0$ . We assume that the initial amplitude of these waves is relatively small. To set the initial position of rolling wave succession, we consider section  $[x_*, x_* + X]$ , corresponding to  $[a(0), c(0)] \subset [x_*, x_* + X]$ , and expand solution (3), limited in this section, to periodic solution  $h = \bar{h}(x, t)$ , meeting conditions

$$\bar{h}(x, 0) = \tilde{h}(x, 0) \quad \forall x \in [x_*, x_* + X], \quad \bar{h}(x + X, 0) = \bar{h}(x, 0) \quad \forall x \in \mathbf{R}. \quad (5)$$

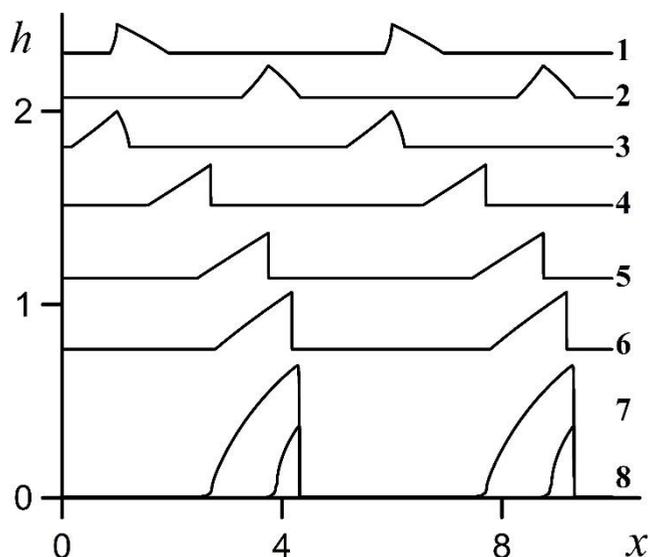
In the case of condensation, i.e., at  $\beta > 0$ , time evolution of infinite succession of waves of type (3) rolling in the residual layer  $H(t, \eta, \beta)$  at  $\eta_l = \eta_r = \eta$ , was considered in [4].

### 3 Results of the calculations

The solution to Cauchy problem is presented in the figure for eight successive time points for equation (2) with periodic initial conditions (5). This solution is continuous (lines 1-3 in fig.1.) during period  $(0, \bar{T}_1)$ , where  $\bar{T}_1 = (V_+^2 - \eta^2) / (2\beta)$  is time point, determined from equation  $H(\bar{T}_1, \eta, \beta) = \sqrt{V_+}$ , where residual depth  $H(t, \eta, \beta)$  achieves the lower existence limit of decreasing wave  $h_+$ . At  $t > \bar{T}_1$ , strong discontinuities (lines 4-8 in fig.1.) are formed on the leading edges of rolling waves. During period  $(\bar{T}_1, T(\eta, \beta))$  succession of rolling waves with strong discontinuities on the leading edges falls down in the residual layer, whose depth decreases (lines 4-6 in fig.1.). At  $t = T(\eta, \beta)$ , the residual layer evaporates completely (line 7 in fig.1.), and at  $t > T(\eta, \beta)$ , these waves continue spreading over the dry wall (line 8 in fig.).

### 4 Conclusion

Here, we examine the model problem concerning the flow of a thin liquid layer on a vertical wall with consideration of heat and mass transfer at the interface in the regime of periodic rolling waves. The families of generalized periodic solutions to the rolling wave equation are presented, the evolution of exact solutions is studied.



**Fig. 1.** Solution to the Cauchy problem for equation (2) with periodic initial conditions (5) for eight successive time points  $t = 0, 0.5, 1.0, 1.5, 2.0, 2.35, 2.645, 2.8$  and following values of initial parameters  $\beta = -1, V_+ = 3.0, V_- = 6.0, \eta_l = \eta_r = 2.3, x_0^+ = 4.25, x_0^- = 2.0, x_* = 0.875, X = 5.0$ .

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