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Extension of Some Positive and Linear Operators to Domains with Curved Sides

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Abstract. There are constructed some Cheney-Sharma type operators defined on a square with one curved side. They are extensions of the Cheney-Sharma type operators of second kind, given by E.W. Cheney and A. Sharma in [14], to the case of a curved sided domain. There are constructed the univariate Cheney-Sharma type operators, their product and boolean sum operators and there are studied their properties, their orders of accuracy and the remainders of the corresponding approximation formulas. Finally, there are given some illustrative examples.

1 Introduction

The aim of this paper is to construct some Cheney-Sharma type operators that have some interpolatory properties on a curved sided domain. There will be studied the interpolation properties, the orders of accuracy and the remainders of the corresponding approximation formulas.

Using the interpolation properties of such operators, blending function interpolants can be constructed, that exactly match function on some sides of a rectangular region. Important applications of these blending functions are in finite element method for differential equations problems with Dirichlet boundary conditions or for construction of surfaces which satisfy some given conditions.

2 Related reviews

Starting with the paper [3] of R.E. Barnhill, G. Birkhoff and W.J. Gordon, there have been constructed interpolation operators of Lagrange, Hermite and Birkhoff type, that interpolate the values of a given function or the values of the function and of certain of its derivatives on the boundary of a triangle with straight sides. These operators were applied in computer aided geometric design (see, e.g., [1]-[3], [5]]) and in finite element analysis (see, e.g., [1], [7], [8], [13], [17]-[19], [21], [22]).

In order to match all the boundary information on a curved domain (as in Dirichlet, Neumann or Robin boundary conditions for differential equation problems), there were considered interpolation operators on domains with curved sides (see, e.g., [4], [6], [9]-[12], [15], [16], [20], [21], [23]). Approximation operators on polygonal domains with some curved sides have also important applications especially in finite element method for differential equations with given boundary conditions and in the piecewise generation of surfaces in computer aided geometric design.

3 Methods

Let $m \in \mathbf{N}$ and β a nonnegative parameter. The Cheney-Sharma operators of second kind Q_m:C([0,1])→C([0,1]), introduced in [14], are given by

$$(Q_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f(\frac{i}{m}), \quad (1)$$

$$q_{m,i}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}.$$

Remark. 1) Notice that for β=0, the operator Q_m becomes the Bernstein operator.

2) In [26], it has been proved that the Cheney-Sharma operator Q_m interpolates a given function at the endpoints of the interval.

3) In [14] and [26], it has been proved that the Cheney-Sharma operator Q_m reproduces the constant and the linear functions, so its degree of exactness is 1 (denoted dex(Q_m)=1).

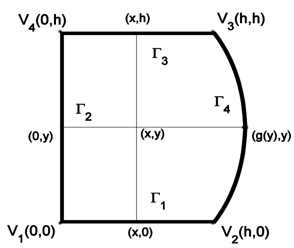

Figure 1. The square D_h.

For $h \in R_+$, let D_h be the square with one curved side having the vertices $V_1=(0,0)$, $V_2=(h,0)$, $V_3=(h,h)$ and $V_4=(0,h)$, three straight sides, Γ_1, Γ_2, along the coordinate axes and Γ_3 parallel to axis Ox, and the curved side Γ_4, that is defined by the function g, such that g(h)=g(0)=h (See Fig. 1).

4 Main results

4.1 Univariate operators

Let F be a real-valued function defined on D_h and (0,y), (g(y),y), respectively, (x,0), (x,h) be the points in which the parallel lines to the coordinate axes, passing through the point $(x,y) \in D_h$, intersect the sides Γ_2, Γ_4, respectively Γ_1 and Γ_3. We consider the uniform partitions of the intervals [0,g(y)] and [0,h], with $y \in [0,h]$:

$$\Delta_m^x = \{\frac{i}{m} g(y) \mid i=0,...,m\}$$

and

$$\Delta_n^y = \{\frac{j}{n} h \mid j=0,...,n\}.$$

For $m,n \in \mathbf{N}$, $b, \beta \in \mathbf{R}_+$, we consider the following extensions of the Cheney-Sharma operator given in (1):

$$(Q_m^x F)(x,y) = \sum_{i=0}^{m} q_{m,i}(x,y) F(\frac{i}{m} g(y), y), \qquad (2)$$

$$(Q_n^y F)(x,y) = \sum_{j=0}^{m} q_{n,j}(x,y) F(x, \frac{j}{n} h),$$

with

$$q_{m,i}(x) = \binom{m}{i} \frac{\frac{x}{g(y)}(\frac{x}{g(y)} + i\beta)^{i-1}(1-\frac{x}{g(y)})}{(1+m\beta)^{m-1}} \cdot [1-\frac{x}{g(y)} + (m-i)\beta]^{m-i-1}$$

$$q_{n,j}(x) = \binom{n}{j} \frac{\frac{y}{h}(\frac{y}{h} + jb)^{j-1}(1-\frac{y}{h})[1-\frac{y}{h}+(n-j)b]^{n-j-1}}{(1+m\beta)^{m-1}}.$$

Remark. As Cheney-Sharma operator of second kind interpolates a given function at the endpoints of the interval, we may use the operators Q_m^x and Q_n^y as interpolation operators on D_h.

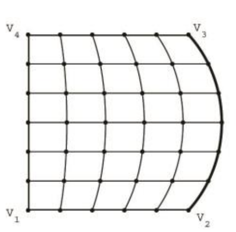

Figure 2.

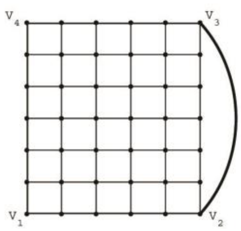

Figure 3.

In Figures 2 and 3 we plot the points $(\frac{i}{m} g(y), y)$, i=0,...,m, and respectively, $(x, \frac{j}{n} h)$, j=0,...,n, for x, $y \in [0,h]$, and m=5, n=6.

Theorem. If F is a real-valued function defined on D_h then

(i) Q_m^x F=F on $\Gamma_2 \cup \Gamma_4$,

(ii) Q_n^y F=F on $\Gamma_1 \cup \Gamma_3$.

Proof. (i) We may write

$$(Q_m^x F)(x,y) = \frac{1}{(1+m\beta)^{m-1}} \Bigg\{ (1-\frac{x}{g(y)}) \left[1-\frac{x}{g(y)} + m\beta\right]^{m-1} \cdot F(0,y) + \frac{x}{g(y)}(1-\frac{x}{g(y)}) \sum_{i=0}^{m} \binom{m}{i} (\frac{x}{g(y)} + i\beta)^{i-1} \qquad (3)$$
$$\cdot \left[1-\frac{x}{g(y)} + (m-i)\beta\right]^{m-i-1} F(\frac{i}{m} g(y), y) + \frac{x}{g(y)}(\frac{x}{g(y)} + m\beta)^{m-1} F(g(y),y) \Bigg\}.$$

Considering (3) we may easily prove that

$$(Q_m^x F)(0,y) = F(0,y)$$

$$(Q_m^x F)(g(y),y) = F(g(y),y).$$

(ii) The proof follows the same steps as for (i).

Theorem. The operators Q_m^x and Q_n^y have the following orders of accuracy:

(i) $(Q_m^x e_{ij})(x,y) = x^i y^j$, $i = 0,1; j \in N$;

(ii) $(Q_n^y e_{ij})(x,y) = x^i y^j$, $i \in N;\ j = 0,1$, where $e_{ij}(x, y)=x^i y^j$, $i, j \in N$.

Proof. The proof follows by the property $dex(Q_m)=1$.

We consider the approximation formula

$$F = Q_m^x F + R_m^x F,$$

where $R_m^x F$ denotes the approximation error.

Theorem. If $F(\cdot, y) \in C^2[0,g(y)]$ then

$$(R_m^x F)(x,y) = \frac{1}{2} F^{(2,0)}(\xi, y)[x^2 - (Q_m^x e_{20})(x,y)], \text{ for } \xi \in [0,g(y)].$$

Proof. Taking into account that $dex(Q_m^x) = 1$, and applying Peano's theorem (see, e.g., [25]), it follows

$$(R_m^x F)(x,y) = \int_0^{g(y)} K_{20}(x,y;s) F^{(2,0)}(s,y) ds,$$

where

$$K_{20}(x,y;s) = (x-s)_+ - \sum_{i=0}^{m} q_{m,i}(x,y)(\frac{i}{m} g(y) - s)_+.$$

For a given $\nu \in \{1,...,m\}$, one denotes by $K_{20}^{\nu}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m}, \nu\frac{g(y)}{m}\right]$, i.e.,

$$K_{20}^{\nu}(x,y;\nu) = (x-s)_{+} - \sum_{i=\nu}^{m} q_{m,i}(x,y)(\frac{i}{m}g(y)-s),$$

whence,

$$K_{20}^{\nu}(x,y;s) = \begin{cases} x-s-\sum_{i=\nu}^{m} q_{m,i}(x,y)(\frac{i}{m}g(y)-s), & s<x \\ -\sum_{i=\nu}^{m} q_{m,i}(x,y)(\frac{i}{m}g(y)-s), & s \geq x. \end{cases}$$

It follows that $K_{20}^{\nu}(x,y;s) \leq 0$, for $s \geq x$.

For s<x we have

$$K_{20}^{\nu}(x,y;s) = x-s-\sum_{i=0}^{m} q_{m,i}(x,y)(\frac{i}{m}g(y)-s) + \sum_{i=0}^{\nu-1} q_{m,i}(x,y)(\frac{i}{m}g(y)-s).$$

Applying the previous Theorem, we get

$$\sum_{i=0}^{m} q_{m,i}(x,y)(\frac{i}{m}g(y)-s) = (Q_m^x e_{10})(x,y) - s(Q_m^x e_{00})(x,y) = x-s,$$

and it follows that

$$K_{20}^{\nu}(x,y;s) = \sum_{i=0}^{\nu-1} q_{m,i}(x,y)(\frac{i}{m}g(y)-s) \leq 0.$$

So, $K_{20}^{\nu}(x,y;\cdot) \leq 0$, for any $\nu \in \{1,...,m\}$, i.e., $K_{20}(x,y;s) \leq 0$, for $s \in [0,g(y)]$.

By Mean Value Theorem, one obtains $(R_m^x F)(x,y) = F^{(2,0)}(\xi,y)\int_0^{g(y)} K_{20}(x,y;s)ds,\ 0 \leq \xi \leq g(y),$ with

$$\int_0^{g(y)} K_{20}(x,y;s)ds = \frac{1}{2}[x^2 - (Q_m^x e_{20})(x,y)].$$

Remark. Analogous results are obtained for the remainder $R_n^y F$ of the formula

$$F = Q_n^y F + R_n^y F.$$

4.2 Product operators

Let

$$(P_{mn}^1 F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y) q_{n,j}(i\frac{g(y)}{m}, y) F(i\frac{g(y)}{m}, j\frac{h}{n}),$$

$$(P_{nm}^2 F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,j\frac{h}{n}) q_{n,j}(x,y) F(\frac{i}{m}g(j\frac{h}{n}), j\frac{h}{n}).$$

The nodes of $(P_{mn}^1 F)(x,y)$ and $(P_{nm}^2 F)(x,y)$ are given in Figures 4 and 5.

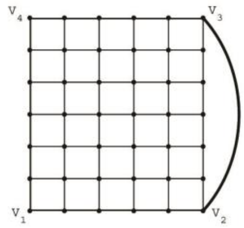

Figure 4. The nodes of $(P_{mn}^1 F)(x,y)$.

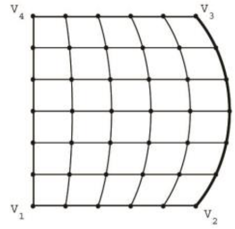

Figure 5. The nodes of $(P_{nm}^2 F)(x,y)$.

Theorem. If F is a real-valued function defined on D_h, then

(i) $(P_{mn}^1 F)(V_i) = F(V_i), \quad i = 1,...,4$

(ii) $(P_{nm}^2 F)(V_i) = F(V_i), \quad i = 1,...,4$.

Proof. The proof follows by a straightforward computation.

4.3 Boolean sum operators

We consider the Boolean sums of the operators Q_m^x and Q_n^y, i.e.,

$$S_{mn}^1 = Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y,$$

$$S_{nm}^2 = Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x.$$

Theorem. If F is a real-valued function defined on D_h, then

$$S_{mn}^1 F\big|_{\partial D_h} = F\big|_{\partial D_h}$$

and

$$S_{nm}^2 F\big|_{\partial D_h} = F\big|_{\partial D_h}.$$

Proof. The proof follows by a straightforward computation.

5 Numerical examples

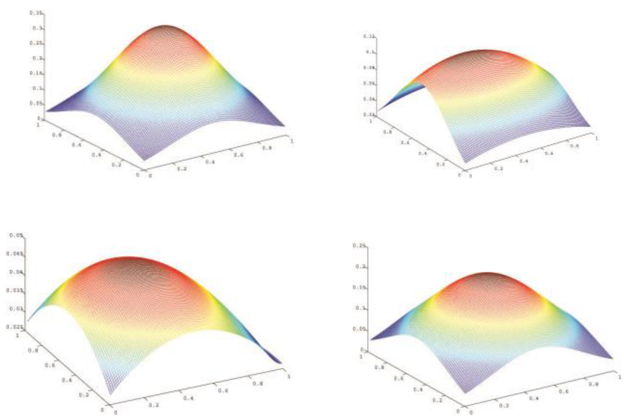

Figure 6. The graphs of F, $Q_m^x F$, $P_{mn}^1 F$, $S_{mn}^1 F$.

We consider the function:

Gentle: $F(x,y)=\frac{1}{3}\exp[-\frac{81}{16}((x-0.5)^2+(y-0.5)^2)]$, generally used in the literature, (see, e.g., [24]). In Figure 6 we plot the graphs of F, $Q_m^x F$, $P_{mn}^1 F$, $S_{mn}^1 F$, on D_h, considering *h=1, m=5, n=6,* $\beta=1$.

Table 1 contains the maximum approximation errors for approximation by $Q_m^x F$, $P_{mn}^1 F$, $S_{mn}^1 F$, considering *h=1, m=5, n=6,* $\beta=1$.

Table 1. The maximum approximation errors.

The approximant	The error
$Q_m^x F$	0.2205
$P_{mn}^1 F$	0.2858
$S_{mn}^1 F$	0.1277

6 Conclusions

The univariate operators obtained here, Q_m^x and Q_n^y, interpolate a function F:$D_h \to \mathbf{R}$ on two sides of the domain D_h, their products, P_{mn}^1 and P_{nm}^2, interpolate on the vertices of D_h, and their boolean sum operators interpolate on the entire frontier of D_h. There are generated the interpolation formulas and there are given the expressions of the corresponding remainders using Peano's theorem. The good approximation properties could be seen in the figures and in the table with errors presented in the previous section. Using the interpolation properties of these operators, blending function interpolants, that exactly match function on some sides of a curved domain region, can be constructed. Important applications of these blending functions are in finite element method for differential equations problems with Dirichlet boundary conditions or for construction of surfaces which satisfy some given conditions.

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