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A One-dimensional Ubiquitiformal Constitutive Model for a Bi-material Rod

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Abstract. A one-dimensional linear elastic ubiquitiformal constitutive model for a bi-material rod under uniaxial tension is developed, in which, the explicit analytical expressions for both the effective elastic modulus and the displacement distribution are obtained. The calculated results of the effective elastic modulus for concrete are found to be in good agreement with previous experimental data.

1. Introduction

As is well known, over the past decades, researchers have been concerned with the constitutive relationship of composites, and it is of special significance to obtain the macroscopic mechanical properties from the mesostructure. Concrete is one kind of widely used construction material in practice, and several constitutive models have been proposed, such as the Hirsch model [1] and the Counto model [2]. Moreover, pioneered by the work of Mandelbrot 3, fractal has also been extensively used to describe the constitutive properties of materials. For example, using the renormalization group procedure to a fractal cantor set, Carpinteri and Cornetti 4 defined a fractal strain and stress and then obtained the constant displacement and the constant elastic modulus for a one-dimensional stretched fractal bar under uniaxial tensile loads. Recently, Davey and Alonso Rasgado 5, *via* constructing a map between a pre-fractal cantor bar and a continuous one-dimensional bar, obtained the finite displacement and the finite strain energy for a one-dimensional bi-material rod. However, both the displacement distributions presented in the two papers are not continuous in the internal boundaries between the matrix and the aggregate materials, and, moreover, both the solutions implied the rigid aggregates and the matrix with vanished length and elastic modulus, which is not the case in practical materials. To avoid such an intrinsic difficulty in fractal application, a new concept of ubiquitiform is introduced by Ou et al.6. A ubiquitiform is defined as a finite order self-similar (or self-affine) physical configuration constructed usually by a finite iterative procedure.

In this paper, based on the concept of ubiquitiform, a generalized ubiquitiformal ternary Cantor set is proposed to describe a bi-material rod, and the explicit analytical expressions for both the effective elastic modulus and the displacement distribution can then be obtained. Moreover, the calculated results of the effective elastic modulus for concrete are

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found to be in good agreement with the previous experimental data.

2. Ubiquitiformal Constitutive Model for a Bi-material Rod

In general, self-similarity of a physical object is available only in a finite scale range of $[\delta_{min}, \delta_{max}]$, where δ_{min} and δ_{max} are the lower and the upper bound to scale invariance, respectively 6. A generalized ubiquitiformal ternary Cantor set is defined as that constructed from an initial element though a deletion operation, in which the initial element $[0, \delta_{max}]$ is divided into p parts and the middle p–2 part is deleted. An N-th order generalized ubiquitiformal ternary Cantor set is obtained after N recursive procedures (Fig.1). Thus, the complexity of the N-th order generalized ubiquitiformal ternary Cantor set is

$$D = \ln 2/\ln p \,. \qquad (1)$$

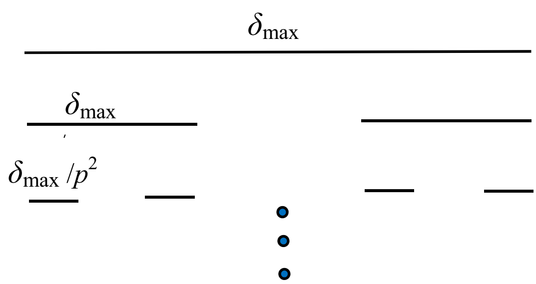

Fig.1. The N-th order generalized ubiquitiformal ternary Cantor set

The N-th order generalized ubiquitiformal ternary Cantor set is used to describe both the distribution of the matrix and the aggregate elements in the bi-material rod. As shown in Fig.2, the element I (remaining parts) includes only matrix, and element II (removed parts) both the matrix and the aggregate.

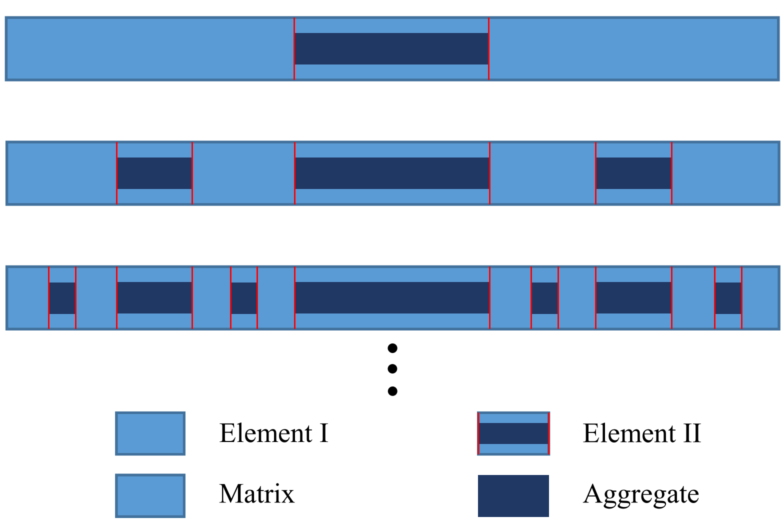

Fig.2. The distribution of the matrix and aggregate material in the bi-material rod

To determine the iteration number N, according to Li et al. 7, there is

$$(1/p)^N = \delta_{\min}/\delta_{\max} \quad . \tag{2}$$

Eliminating p from Eq.(1) And Eq.(2) Gives

$$N = -D \ln\left(\delta_{\min}/\delta_{\max}\right)/\ln 2 \,. \tag{3}$$

For convenience, from Eq.(2) and Eq.(3), the length ratio of the Cantor set of the basic ubiquitiformal cell $(2/p)^N$ can be written as

$$(2/p)^N = \left(\delta_{\min}/\delta_{\max}\right)^{(1-D)} \tag{4}$$

Denoting elastic modulus of the matrix and the aggregate materials as E_m and E_a, respectively, and the elastic modulus of element I and II as E_{I} (= E_m) and E_{II}. A bi-material rod, with cross-sectional area A, is subjected to the force F. The cross section areas of the aggregate and matrix in element II are denoted as A_a and $A_m = A - A_a$, respectively. By the continuity of material, the effect elastic modulus of the element II E_{II} can be obtained as

$$E_{\mathrm{II}} = E_m + \left(E_a - E_m\right) A_a / A \,. \tag{5}$$

To determine A_a, by using the volume ratio r_a of the aggregate in the rod, there is

$$V_a = \left[1 - \left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}\right]\delta_{\max} A_a = r_a \delta_{\max} A \tag{6}$$

Substituting Eq.(3) into Eq.(2) reaches

$$E_{\mathrm{II}} = E_m + r_a\left(E_a - E_m\right)\Big/\left[1 - \left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}\right]. \tag{7}$$

The displacement distribution in the bi-material rod can then be obtained. Denote the displacement of an arbitrary point s in the *N-th* order basic cell as $u^N(s)$ (Fig.2), and the initial lengths of element I and II in the interval [0, s] as $\Delta_{\mathrm{I}}(s)$ and $\Delta_{\mathrm{II}}(s) = s - \Delta_{\mathrm{I}}(s)$, respectively. Under a certain tensile load, $u^N(s)$ can be expressed by

$$u^N(s) = \Delta_{\mathrm{I}}(s) F/\left(E_{\mathrm{I}} A\right) + \left[s - \Delta_{\mathrm{I}}(s)\right] F/\left(E_{\mathrm{II}} A\right) \quad . \tag{8}$$

where

$$\Delta_{\mathrm{I}}(s) = \int_0^s \mu_{\mathrm{I}}(\eta) d\eta \qquad \mu_{\mathrm{I}}(\eta) = \begin{cases} 1 & s \in element \ \mathrm{I} \\ 0 & s \in element \ \mathrm{II} \end{cases} . \tag{9}$$

Substituting Eq.(2) into Eq.(5), there is

$$u^N(s) = \frac{F}{E_m A}\left[\int_0^s \mu_{\mathrm{I}}(\eta) d\eta + \frac{s - \int_0^s \mu_{\mathrm{I}}(\eta) d\eta}{1 + \dfrac{r_a\left(E_a/E_m - 1\right)}{1 - \left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}}}\right]. \tag{10}$$

and then the total extension of the *N-th* order basic ubiquitiformal cell Δu^N is

$$\Delta u^N = \frac{F\delta_{\max}}{E_m A}\left\{\left(\delta_{\min}/\delta_{\max}\right)^{(1-D)} + \frac{1-\left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}}{1+r_a\left(E_a/E_m-1\right)\Big/\left[1-\left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}\right]}\right\}. \quad (11)$$

Denoting the effective elastic modulus of the N-th basic ubiquitiformal cell as E_{eff}, the total extension can be expressed as

$$\Delta u^N = \frac{F\delta_{\max}}{AE_{eff}}. \quad (12)$$

and hence, from Eq.(8) and Eq.(9), one obtains

$$E_{eff} = E_m \Bigg/ \left\{\left(\delta_{\min}/\delta_{\max}\right)^{(1-D)} + \frac{1-\left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}}{1+r_a\left(E_a/E_m-1\right)\Big/\left[1-\left(\delta_{\min}/\delta_{\max}\right)^{(1-D)}\right]}\right\}. \quad (13)$$

3. Numerical Results

To verify the availability of Eq.(13), calculation for the concrete experiment [8] is carried out. With the aggregate gradation given in the experiment, the aggregate mass distribution function can be fitted as $W(d) = (d/d_{\max})^{0.68}$, where d is the diameter of aggregates, and maximum diameter of the aggregates $d_{\max}$ = 19mm. For a represent volume, take the upper bound to scale invariance $\delta_{\max} = 5d_{\max}$. According to Li [9], $\delta_{\min} = 221.38 \times f_t^{-3.24}$, where f_t is the tensile strength of the concrete, and the unit of $\delta_{\min}$ and f_t are μm and MPa, respectively. Finally, the complexity D can be obtained by the aggregate mass distribution function and the aggregate volume ratio 10.

Both the calculated numerical results of E_{eff} and the experimental data of concrete with four different aggregate volume ratio as well as the material properties are listed in Table 1, where E represents the experimental value of the effective elastic modulus, and E_r is the relative error of E_{eff} with respect to E.

TABLE 1. THE EXPERIMENTAL DATA 8 AND THE NUMERICAL RESULTS OF ELASTIC MODULUS OF CONCRETE

r_a (%)	f_t MPa	$\delta_{\min}$ (μm)	$\delta_{\max}$ (mm)	E_m (GPa)	E_a (GPa)	D	p	N	E_{eff} (GPa)	E (GPa)	E_r (%)
20	1.84	30.69	95	11.6	74.5	0.97	2.04	11	14.11	17.8	–20.73
40	1.86	29.63	95	11.6	74.5	0.93	2.11	11	17.91	21.4	–16.31
60	2.38	13.33	95	11.6	74.5	0.86	2.24	12	27.43	29.0	–5.41
80	2.78	8.06	95	11.6	74.5	0.72	2.62	13	48.58	41.3	+17.63

4. Conclusion

A ubiquitiformal constitutive model is proposed, in which the effects of both the aggregate volume ratio and the aggregate gradation on the elastic modulus are taken into account, and the explicit analytical expressions for the effective elastic modulus and the displacement distribution are obtained. The calculated numerical results of the effective elastic modulus for concrete with different aggregate volume ratios are found to be in good agreement with previous experimental data. Moreover, unlike the corresponding fractal displacement distribution, the ubiquitiformal one is continuous at the internal boundaries between the matrix and the aggregates.

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