Representations for the Drazin inverses of block matrices

Li Guo[1,2], Jie Wang[2], Wenming Du[2]

[1] *Department of Mathematics, Southeast University, Nanjing, china*
[2] *College of Mathematics, Beihua University, Jilin, china*

Abstract. In this paper, we investigate representations of the Drazin inverse of a 2×2 block matrix. The Drazin inverse of a matrix is very important in various applied mathematical fields like machinery and automation, singular differential equations . We give a explicit representation of the Drazin inverse of a block matrix.

Keywords: Block matrix; Drazin inverse; Index

1. Introduction

Let $C^{n\times n}$ denote the set of $n\times n$ complex matrices. The smallest nonnegative integer k such that $rank(A^{k+1}) = rank(A^k)$, denoted by $ind(A)$, is called the index of A. The Drazin inverse is the unique matrix $A^d \in C^{n\times n}$ satisfying

$$AA^d = A^d A,\ A^d AA^d = A^d,\ A^{k+1}A^d = A^k,$$

where $k = ind(A)$. We denote by $A^\pi = I - AA^d$. For $A \in C^{n\times n}$ we define $A^0 = I_n$, the identity of order n, even if $A = 0$.

The Drazin inverse of a matrix is very important in various applied mathematical fields like machinery and automation, singular differential equations, singular difference equations, Markov chains, iterative methods and so on[1,2]. We introduce briefly an application of the Drazin inverse of a block matrix[2].

In this paper, representations for the Drazin inverse of 2×2 block matrix M, under the following conditions that $BD = 0,\ D^\pi CB = 0,\ D^\pi CAB = 0,\ D^\pi CA^2 = 0$ and so several results are generalized.

Lemma 1.1 [3]Let $A \in C^{m\times n}$ and $B \in C^{n\times m}$, then $(AB)^d = A[(BA)^2]^d B$.

Lemma 1.2 Let $A \in C^{m\times m}$, then $(A^2)^d = (A^2)^2$.

Lemma 1.3 (see [5],) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a 2×2 block complex matrix, where $A \in C^{m\times m}$ and $D \in C^{n\times n}$ with $ind(A) = r,\ ind(D) = s$, if $BC = 0,\ BD = 0$ then

$$M^d = \begin{pmatrix} A^d & (A^d)^2 B \\ \sum_0 & D^d + \sum_1 B \end{pmatrix},$$

and $k = 0,1$,

$$\sum_k = (D^d)^2 \sum_{i=0}^{r-1} (D^d)^{i+k} CA^i A^\pi + D^\pi \sum_{i=0}^{s-1} D^i C (A^d)^{i+k} (A^d)^2 - \sum_{i=0}^{k} (D^d)^{i+1} C (A^d)^{k-i+1}.$$

with $ind(M) \le ind(A) + ind(D) + 1$.

Lemma 1.4 (see [5],) Let $P, Q \in C^{n\times n}$, where $ind(P) = r,\ ind(Q) = s$, if $PQP = 0,\ PQ^2 = 0$ then

$$(P+Q)^d = Q^\pi \sum_{i=0}^{s-1} Q^i \left(P^D\right)^{i+1} + \sum_{i=0}^{r-1} \left(Q^D\right)^{i+1}\left(P^i\right)P^\pi + Q^\pi \sum_{i=0}^{s-1} Q^i \left(P^D\right)^{i+2} Q$$
$$+\sum_{i=0}^{r-2}\left(Q^D\right)^{i+3}\left(P^{i+1}\right)P^\pi Q Q^D P^D Q - \left(Q^D\right)PP^D Q.$$

2. Main Results

Theorem 2.1 Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a 2×2 block complex matrix, where $A \in C^{m\times m}$ and $D \in C^{n\times n}$ with $ind(A)=r$, $ind(D)=s$, if

$$BD=0,\ D^\pi CB=0,\ D^\pi CAB=0,\ D^\pi CA^2=0,$$

then

$$M^D = \begin{pmatrix} A^D + \left(A^D\right)^3 BC + \left(A^D\right)^4 BCA & \left(A^D\right)^2 B + \left(A^D\right)^4 \mathrm{BCB} \\ \psi_0 + \psi_0 \left(A^D\right)^2 BC + D^D \psi_1 BC + \Delta DD^D C & D^D + \psi_1 B + \Delta BD^\pi \end{pmatrix}, \quad (2.1)$$

where

$$\Delta = \psi_0 \left(A^D\right)^3 BC + D^D \psi_0 \left(A^D\right)^2 BC + \psi_0 D^D \psi_1 BC,$$
$$\psi_k = \sum_{i=0}^{r-1}\left(D^D\right)^{i+k+2} CA^i A^\pi - \sum_{i=0}^{k}\left(D^D\right)^{i+1} C\left(A^D\right)^{k-i+1},\ k=0,1.$$

Proof. Let $M = P+Q$, where

$$P = \begin{pmatrix} 0 & BDD^D \\ D^\pi C & 0 \end{pmatrix},\ Q = \begin{pmatrix} A & BD^\pi \\ DD^D C & D \end{pmatrix}.$$

By $BD=0$, $D^\pi CB=0$, then

$$PQP = \begin{pmatrix} 0 & BDD^\pi \\ D^\pi C & 0 \end{pmatrix}\begin{pmatrix} A & BD^\pi \\ DD^D C & D \end{pmatrix}\begin{pmatrix} 0 & BDD^D \\ D^\pi C & 0 \end{pmatrix}$$
$$= \begin{pmatrix} BDD^D \mathrm{C} & BD^2 D^D \\ D^\pi CA & D^\pi CBD^\pi \end{pmatrix}\begin{pmatrix} 0 & BDD^D \\ D^\pi C & 0 \end{pmatrix}$$
$$=0.$$

Since $D^\pi CB=0$, $D^\pi CAB=0$, $D^\pi CA^2=0$, we have

$$PQ^2 = \begin{pmatrix} 0 & BDD^D \\ D^\pi C & 0 \end{pmatrix}\begin{pmatrix} A & BD^\pi \\ DD^D C & D \end{pmatrix}\begin{pmatrix} A & BD^\pi \\ D^D DC & D \end{pmatrix}$$
$$= \begin{pmatrix} BDD^D \mathrm{C} & BD^2 D^D \\ D^\pi CA & D^\pi CBD^\pi \end{pmatrix}\begin{pmatrix} A & BD^\pi \\ DD^D C & D \end{pmatrix}$$
$$= \begin{pmatrix} BDD^D CA + BD^2 D^D C & BDD^D CBD^D + BD^3 D^D \\ D^\pi CA^2 & D^\pi CABD^\pi + D^\pi CBD^\pi \mathrm{D} \end{pmatrix}$$
$$=0.$$

By $P^2=0$, we get that $P^D=0$, $P^\pi = I$. Noting that $BD=0$, by Lemma 1.3, we have

$$Q^D = \begin{pmatrix} A^D & \left(A^D\right)^2 B \\ \psi_0 & D^D + \psi_1 B \end{pmatrix},$$

where

$$\psi_k=\sum_{i=0}^{r-1}\left(D^D\right)^{i+k+2}CA^iA^\pi-\sum_{i=0}^{k}\left(D^D\right)^{i+1}C\left(A^D\right)^{k-i+1},\quad k=0,1.$$

Since $PQP=0$, $PQ^2=0$, By Lemma 1.4, we get

$$\left(P+Q\right)^D=Q^\pi\sum_{n=0}^{k-1}Q^i\left(P^D\right)^{i+1}+\sum_{i=0}^{m-1}\left(Q^D\right)^{i+1}P^iP^\pi+Q^\pi\sum_{i=0}^{k-1}Q^i\left(P^D\right)^{i+2}Q$$
$$+\sum_{i=0}^{m-2}\left(Q^D\right)^{i+3}P^{i+1}P^\pi Q-Q^DP^DQ-\left(Q^D\right)^2PP^DQ,$$

where $k=ind\left(Q\right)\le r+s+1$, $m=ind\left(P\right)=2$, ,

By $P^D=0$, $P^\pi=I$, then

$$\begin{aligned}\left(P+Q\right)^D&=\sum_{i=0}^{m-1}\left(Q^D\right)^{i+1}P^i+Q^\pi\sum_{i=0}^{m-2}\left(Q^D\right)^{i+3}P^{i+1}Q\\&=Q^D+\left(Q^D\right)^2P+\left(Q^D\right)^3PQ\end{aligned},$$

$$\begin{aligned}\left(Q^D\right)^2P&=\begin{pmatrix}A^D&\left(A^D\right)^2B\\\psi_0&D^D+\psi_1\mathrm{B}\end{pmatrix}\begin{pmatrix}\left(A^D\right)^2BD^\pi C&0\\\psi_1BD^\pi C&0\end{pmatrix}\\&=\begin{pmatrix}\left(A^D\right)^3BC+\left(A^D\right)^2\psi_1BC&0\\\psi_0\left(A^D\right)^2BC+D^D\psi_1BC+\psi_1B\psi_1BC&0\end{pmatrix}.\end{aligned}$$

By $BD=0$, $BD^D=0$, $B\psi_1=0$, so we have

$$\left(Q^D\right)^2P=\begin{pmatrix}\left(A^D\right)^3BC&0\\\psi_0\left(A^D\right)^2BC+D^D\psi_1BC&0\end{pmatrix}.$$

By $B\psi_0=0$, $B\psi_1=0$, $BD^D=0$, we get

$$\begin{aligned}\left(Q^D\right)^3P&=\begin{pmatrix}A^D&\left(A^D\right)^2B\\\psi_0&D^D+\psi_1\mathrm{B}\end{pmatrix}\begin{pmatrix}\left(A^D\right)^3BC&0\\\psi_0\left(A^D\right)^2BC+D^D\psi_1BC&0\end{pmatrix}\\&=\begin{pmatrix}\left(A^D\right)^4BC&0\\\psi_0\left(A^D\right)^3BC+D^D\psi_0\left(A^D\right)^2BC+\psi_0D^D\psi_1BC&0\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\left(Q^D\right)^3PQ&=\begin{pmatrix}\left(A^D\right)^4BC&0\\\psi_0\left(A^D\right)^3BC+D^D\psi_0\left(A^D\right)^2BC+\psi_0D^D\psi_1BC&0\end{pmatrix}\begin{pmatrix}A&BD^\pi\\DD^DC&\mathrm{D}\end{pmatrix}\\&=\begin{pmatrix}\left(A^D\right)^4BCA&\left(A^D\right)^4BCB\\\Delta DD^D\mathrm{C}&\Delta\mathrm{BD}^\pi\end{pmatrix}.\end{aligned}$$

where

$$\Delta=\psi_0\left(A^D\right)^3BC+D^D\psi_0\left(A^D\right)^2BC+\psi_0D^D\psi_1BC.$$

So we have

$$
\begin{aligned}
(P+Q)^D &= Q^D + \left(Q^D\right)^2 P + \left(Q^D\right)^3 PQ \\
&= \begin{pmatrix} A^D + \left(A^D\right)^3 BC + \left(A^D\right)^4 BCA & \left(A^D\right)^2 B + \left(A^D\right)^4 BCB \\ \psi_0 + \psi_0\left(A^D\right)^2 BC + D^D\psi_1 BC + \Delta DD^D C & D^D + \psi_1 B + \Delta BD^\pi \end{pmatrix}.
\end{aligned}
$$

This result generalizes the result under the condition $BD = 0$, $D^\pi CB = 0$, $D^\pi CA = 0$ in [4]

References

[1] Campbell S L, Meyer C D, Jr. Generalized Inverses of Linear Transformations. London: Pitman Press, 1979.

[2] Campbell S L. The Drazin inverse and systems of second order linear differential equations. Linear Multili -near Algebra, Vol. 14 (1983), 195-198.

[3] D. S. Djordjevic, Y. Wei,, Additive results for the generalized Drazin inverse, J. Aust. Math. Soc. Vol. 73 (2002), 115 -126,.

[4] Dopazo E, Matrinez-Serrano M F. Further results on the representation of the Drazin inverse of a 2×2 block matrix. Linear Algebra Appl, Vol. 43 (2010), 1896-1904.

[5] Li Guo, Xiankun Du. Representations for the Drazin inverses of 2×2 block matrices. Applied Mathematics and Computation, Vol. 217 (2010), 2833-2842.