Representations for the Drazin inverses of block matrices

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Abstract. In this paper, we investigate representations of the Drazin inverse of a $2 \times 2$ block matrix. The Drazin inverse of a matrix is very important in various applied mathematical fields like machinery and automation, singular differential equations. We give a explicit representation of the Drazin inverse of a block matrix.

Keywords: Block matrix; Drazin inverse; Index

1. Introduction

Let $C^{m\times n}$ denote the set of $n \times n$ complex matrices. The smallest nonnegative integer $k$ such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$, denoted by $\text{ind}(A)$, is called the index of $A$. The Drazin inverse is the unique matrix $A^d \in C^{m\times n}$ satisfying

$$A^d = A^d A, A^d AA^d = A^d, A^{k+1}A^d = A^k,$$

where $k = \text{ind}(A)$. We denote by $A^d = I - AA^d$. For $A \in C^{m\times n}$ we define $A^0 = I_n$, the identity of order $n$, even if $A = 0$.

The Drazin inverse of a matrix is very important in various applied mathematical fields like machinery and automation, singular differential equations, singular difference equations, Markov chains, iterative methods and so on[1,2]. We introduce briefly an application of the Drazin inverse of a block matrix[2].

$$\sum_{k} = (D^d)^{i-1} C A^d A^d + D^d \sum_{i=0}^{k} D^d C(A^d) \left(A^d\right)^{i-1} C(A^d)^{-1}.$$

with $\text{ind}(M) \leq \text{ind}(A) + \text{ind}(D) + 1$.

Lemma 1.4 (see [5],) Let $P, Q \in C^{m\times n}$, where

In this paper, representations for the Drazin inverse of $2 \times 2$ block matrix $M$, under the following conditions that $BD = 0$, $D^2CB = 0$, $D^2CA^2 = 0$ and several results are generalized.

Lemma 1.1 [3] Let $A \in C^{m\times n}$ and $B \in C^{m\times n}$, then $(AB)^d = A[(BA)^2]^d B$.

Lemma 1.2 Let $A \in C^{m\times n}$, then $(A^2)^d = (A^2)^2$.

Lemma 1.3 (see [5],) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a $2 \times 2$ block complex matrix, where $A \in C^{m\times m}$ and $D \in C^{m\times n}$ with $\text{ind}(A) = r$, $\text{ind}(D) = s$ if $BC = 0$, $BD = 0$ then

$$M^d = \begin{pmatrix} A^d & (A^d)^{i-1} B \\ \sum_{0}^{k} D^d + \sum_{1}^{k} B \end{pmatrix},$$

and $k = 0.1$.

$\text{ind}(P) = r$, $\text{ind}(Q) = s$, if $PQP = 0$, $PQ^2 = 0$ then
\[(P + Q)^i = Q^i \sum_{j=0}^{i-1} Q^j (P^D)^j + \sum_{j=0}^{i-1} (Q^D)^j (P^\sigma)^j + Q^i \sum_{j=0}^{i-1} Q^j (P^D)^j Q^j + \sum_{j=0}^{i-1} (Q^D)^j (P^\sigma)^j PQD + (Q^D)PP^D Q.\]

2. Main Results

**Theorem 2.1** Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be a \(2 \times 2\) block complex matrix, where \( A \in \mathbb{C}^{m \times m} \) and \( D \in \mathbb{C}^{n \times n} \) with \( \text{ind}(A) = r, \ \text{ind}(D) = s \), if

\[
BD = 0, \ D^\sigma CB = 0, \ D^\sigma CAB = 0, \ D^\sigma CA^2 = 0,
\]

then

\[
M^D = \begin{pmatrix} A^D + \left( A^D \right)^k BC + \left( A^D \right)^k BCA \ & \left( A^D \right)^k B + \left( A^D \right)^k BCB \\ \psi_0 + \psi_0 \left( A^D \right)^k BC + B^D \psi_1 BC + \Delta DD^D C + D^D + \psi_1 B + \Delta BD^D \end{pmatrix},
\]

(2.1)

where

\[
\Delta = \psi_0 \left( A^D \right)^k BC + B^D \psi_0 \left( A^D \right)^k BC + \psi_0 \psi_1 BC;
\]

\[
\psi_1 = \sum_{i=0}^{k-1} \left( D^D \right)^i C A^k A^\sigma - \sum_{i=0}^{k-1} \left( D^D \right)^i C \left( A^D \right)^i, \ \text{if} \ k = 0,1.
\]

**Proof.** Let \( M = P + Q \), where

\[
P = \begin{pmatrix} 0 & BDD^0 \\ D^\sigma C & 0 \end{pmatrix}, \ Q = \begin{pmatrix} A & BD^\sigma \\ DD^0 C & D \end{pmatrix}.
\]

By \( BD = 0, \ D^\sigma CB = 0 \), then

\[
PQP = \begin{pmatrix} 0 & BDD^\sigma \\ D^\sigma C & 0 \end{pmatrix} \begin{pmatrix} A & BD^\sigma \\ DD^0 C & D \end{pmatrix} \begin{pmatrix} 0 & BDD^D \\ D^\sigma C & 0 \end{pmatrix} = \begin{pmatrix} 0 & BDD^D \\ D^\sigma C & 0 \end{pmatrix} = 0.
\]

Since \( D^\sigma CB = 0, \ D^\sigma CAB = 0, \ D^\sigma CA^2 = 0 \), we have

\[
PQP^2 = \begin{pmatrix} 0 & BDD^D \\ D^\sigma C & 0 \end{pmatrix} \begin{pmatrix} A & BD^\sigma \\ DD^0 C & D \end{pmatrix} \begin{pmatrix} A & BD^\sigma \\ DD^0 C & D \end{pmatrix} \begin{pmatrix} 0 & BDD^D \\ D^\sigma C & 0 \end{pmatrix} = \begin{pmatrix} 0 & BDD^D CA + BD^2 D^D + BD^D CBD^D + BD^D D^D \\ D^\sigma CA + D^\sigma CBD^D + D^\sigma CBD^D D \end{pmatrix} = 0.
\]

By \( P^2 = 0 \), we get that \( P^D = 0, \ P^\sigma = I. \) Noting that \( BD = 0 \), by Lemma 1.3, we have

\[
Q^D = \begin{pmatrix} A^D & \left( A^D \right)^k B \\ \psi_0 & D^D + \psi_1 B \end{pmatrix},
\]

where
\[
\psi_k = \sum_{i=0}^{k-1} (D^D)^{i+k+2} CA' A^\tau - \sum_{i=0}^{k} (D^D)^{i+1} C (A^D)^{k-i+1}, \quad k = 0.1.
\]

Since \(PQP = 0, \ PQ^2 = 0\), By Lemma 1.4, we get

\[
(P + Q)^D = Q^\tau \sum_{i=0}^{k-1} (P^D)^{i+1} + \sum_{i=0}^{m-1} (Q^D)^{i+1} P^i P^\tau + Q^\tau \sum_{i=0}^{m-2} (P^D)^{i+1} Q^i P^i Q^D Q.
\]

where \(k = \text{ind}(Q) \leq r + s + 1, \ m = \text{ind}(P) = 2r\), By \(P^D = 0, \ P^\tau = I\), then

\[
(P + Q)^D = \sum_{i=0}^{m-1} (Q^D)^{j+1} P^i + Q^\tau \sum_{i=0}^{m-2} (Q^D)^{j+1} P^i Q^D Q.
\]

By \(BD = 0, \ BD^D = 0, \ B\psi_1 = 0\), so we have

\[
(Q^D)^2 P = \begin{pmatrix}
(A^D)^3 BC & 0 \\
\psi_0 (A^D)^2 BC + D^D \psi_1 BC & 0
\end{pmatrix}
\]

By \(B\psi_0 = 0, \ B\psi_1 = 0\), \(BD^D = 0\), we get

\[
(Q^D)^2 P = \begin{pmatrix}
(A^D)^3 BC & 0 \\
\psi_0 (A^D)^2 BC + D^D \psi_1 BC & 0
\end{pmatrix}
\]

and

\[
(Q^D)^2 PQ = \begin{pmatrix}
(A^D)^4 BC & 0 \\
\psi_0 (A^D)^3 BC + D^D \psi_0 (A^D)^2 BC + \psi_0 D^D \psi_1 BC & 0
\end{pmatrix}
\]

where

\[
\Delta = \psi_0 (A^D)^3 BC + D^D \psi_0 (A^D)^2 BC + \psi_0 D^D \psi_1 BC
\]

So we have
\[(P + Q)^D = Q^D + (Q^D)^\dag P + (Q^D)^\dag P Q\]

This result generalizes the result under the condition \( BD = 0, \ D^+ CB = 0, \ D^+ CA = 0 \) in [4]

References