Algorithm Robustness Analysis for the Choice of Optimal Time Delay of Phase Space Reconstruction Based on Singular Entropy Method

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Abstract. The method of Delays is commonly used in state space reconstruction of chaotic time series, and the quality of attractor reconstruction of phase space is sensitive to the choice of embedding parameters, i.e. embedding dimension and time delay. In this paper, singular entropy (SE) is applied to choose the optimal time delay in reconstructed dynamical system, and the comparisons of relevant results are made with the correlation function method by using singular value fraction (SVF). The robustness of singular entropy algorithm is further discussed, and it shows the advantage of the proposed method.

Keywords. singular entropy; phase space reconstruction; the optimal time delay; robustness

0 Introduction

The analysis of chaotic time series has become popular in many fields of science and engineering since the work of Packard [1] and Takens [2] who initially introduced the delay-coordinate embedding technique to reconstruct a smooth copy of the underlying dynamics. According to Takens’ theorem, a scalar time series \( \{x_i\}, i = 1, 2, ..., N \) can be embedded into a \( m \)-dimensional space to form multiple state-space vectors \( X_i \) if the embedding dimension \( m \) is equal or more than twice the topological \( n \), i.e. \( m \geq 2n + 1 \). The reconstructed trajectory \( X \) is given by

\[
X = [X_1, X_2, ..., X_N]^T
\]  

The reconstructed state of the system at each discrete time \( i \) is

\[
X_i = (x_i, x_{i+L}, ..., x_{i+(m-1)L})
\]  

where \( m \) is the embedding dimension, \( L \) the reconstruction time delay defined as the integer multiple of sampling time \( \tau \), and \( N \) the number of vectors in state space which is given as \( N = N - (m - 1)L \).

In fact, the choice of embedding parameters is critical for the quality of attractor reconstruction due to the time series with noise and finite length. An excessively small value of \( L \) results in redundancies between successive delay coordinates with little information, while too large value of \( L \) may cause successive delay coordinates irrelevant so that the reconstruction attractors are no longer representative of true dynamics [3].

Different methods have been proposed to choose the value of \( L \) based on the criteria of sequential correlation and dimensional expansion. Albano suggested using \( L \) as optimal time delay when the autocorrelation function \( R_{xx}(L) \) first drops to 1/e of its initial value [4], and Fraser and Swinney proposed to choose the first minimum value of the mutual information function \( L \) as optimal time delay.
instead [5]. The geometrical concepts of fill factor [6], wavering product [7] and average displacement [8] are also applied to measure the expansion of attractors in the state space. Martinerie considered that it’s more meaningful to estimate \( m \) and \( L \) simultaneously with a time window [9], and Kember proposed the recipe of correlation function termed singular value fraction (SVF) to choose proper time delay [10]. Such methods as high-order correlation [11], nonlinear correlation functions [12], and singular value spectrum entropy [13] are also proposed to overcome the deficiency of autocorrelation function.

In this paper, on the basis of global singular value decomposition (SVD), the concept of singular entropy (SE) [14] is adopted to determine the optimal time delay \( L \). Section 1 presents the definition of singular entropy (SE) and its application in the description of information measure. In Section 2, SE is applied to choose the optimal time delay \( L \) of dynamical reconstruction, where the comparisons of relevant results with SVF method are concerned. Section 3 discusses the influences of the order of singular entropy \( (k \leq m) \), and \( \Delta E_i \), the increment of singular entropy in order \( i \) which is given by

\[
\Delta E_i = -p_i \log p_i
\]

\( p_i \) reflects the proportion of the \( i \)th singular value in the total singular value spectrum that contributes to the global pattern, which is defined as

\[
p_i = \frac{\sigma_i^2}{\sum_{i=1}^{m} \sigma_i^2}
\]

Therefore, singular entropy can be considered as information measure of signals [14], and the example is given as follows.

The following of this paper is concerned with the application of SE for the choice of \( L \), and the comparisons with SVF method for dynamical phase reconstruction is made as well.

2 Applications for the choice of \( L \) with Singular entropy (SE)

The choice of time delay of reconstructed dynamics with singular entropy involves in this section. Three equations of sinusoid and Rössler are employed. \( k \) is chosen to be 1 in the examples and whose influence on \( L \) will be further discussed in Section 4.

2.1 Choice of \( L \) for sinusoid

Fig. 1 shows the curve of \( E(1) \) versus \( L \)

### Equation (4)

\[
(XU)^T XU = \Lambda^2
\]
for time series of \( x(t) = \sqrt{2} \sin \pi t \) with sample time \( \tau = 0.01 \) s and embedding dimension \( m = 2, 3, 4, 5, 7, 9 \) respectively. Larger value of \( E(k) \) implies more information from the first \( k \) singular values of reconstructed dynamics, which is closely related to preferable expansion of attractors.

Figure 1 shows \( E(1) \) versus \( L \) for a time series of \( x(t) = \sqrt{2} \sin \pi t \) with \( m = 2, 3, 4, 5, 7, 9 \). In Fig.1, the moment when first maximum of \( E(1) \) occurs is chosen to represent the optimal time delay denoted as \( \Delta_m \), which \( \Delta_m / m \) is closely related to the time of \( M \) when first minimum occurs and given by

\[
L_m \approx \Delta_m / m \tag{8}
\]

Optimal time window \( \Delta_w \) is defined as

\[
\Delta_w = (m - 1)L_m \approx (m - 1)\Delta_m / m \tag{9}
\]

Similar conclusions can be found in reference [10], while the proposed method here applies singular entropy instead of using linear SVF function.

For comparisons, Fig.2 shows the plots of \( f_{sv}(1) \) versus \( L \) by using the method of SVF for the time series of \( x(t) = \sqrt{2} \sin \pi t \) with \( m = 2, 3, 4, 5, 7, 9 \) respectively.

Figure 2 shows \( f_{sv}(1) \) versus \( L \) for a time series of \( x(t) = \sqrt{2} \sin \pi t \) with \( m = 2, 3, 4, 5, 7, 9 \).

Table 1 shows the variation of \( L_m \) with different \( m \) for time series of \( x(t) = \sqrt{2} \sin \pi t \), and the results are compared with SVF method as well.

<table>
<thead>
<tr>
<th>( L_m )</th>
<th>( \Delta_m )</th>
<th>( \Delta_m / m )</th>
<th>( \Delta_w )</th>
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<tr>
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<td>50</td>
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2.2 Choice of \( L \) for Rössler equations

Consider Rössler system described by the equations

\[
\dot{x} = -y - z , \quad \dot{y} = x + ay , \quad \dot{z} = b + z(x - c),
\]

where \( a = 0.15 \), \( b = 0.20 \), \( c = 10.0 \). The differential equations are solved with numerical integration using a four-order Runge-Kutta algorithm. Initial values are chosen to be \( x_0 = 10 \), \( y_0 = 0 \), \( z_0 = 0 \) and the integral step is fixed at \( \tau = \pi / 100 \).

The first 5000 data samples are discarded.

Figure 3 shows the curve of \( E(1) \) versus \( L \) for Rössler equations using a time series of \( x(t) = \sqrt{2} \sin \pi t \) with embedding dimension \( m = 2, 3, 4, 5, 7, 9 \) respectively.
Table 2 shows the variation of $L_m$ with different $m$ for Rössler equations using a time series of $x(t)$ and resultant comparisons as well.

Table 2

<table>
<thead>
<tr>
<th>$L_m$</th>
<th>$\Delta_m$</th>
<th>$\Delta_m/m$</th>
<th>$\Delta_m$</th>
<th>$m$</th>
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Fig.3 $E(1)$ versus $L$ for Rössler equations using a time series of $x(t)$ with $m = 2, 3, 4, 5, 7, 9$

Fig.4 shows the curve of $f_{sv}(1)$ versus $L$ for Rössler equations using a time series of $x(t)$ with the same parameters as those of Fig.5.

Fig.4 $f_{sv}(1)$ versus $L$ for Rössler equations using a time series of $x(t)$ with $m = 2, 3, 4, 5, 7, 9$

3 The robustness analysis of SE method

The robustness of SE method for the choice of $L$, including the effects of order $k$ and the corresponding signal to noise rates (SNR) of time series, involves in this section.

Fig.5 shows the curves of $E(k)$ versus $L$ for the same time series of (a) $x(t) = \sqrt{2} \sin \pi t$, and (b) Rössler equation respectively as mentioned in Section 2 with fixed $m = 7$ and the variation of $k$ from 1 to 7.

From Fig.5, the choice of larger $k$ fails to reach satisfactory results, and therefore the smaller one is recommended instead.

To examine the sensitivity of SE method to
the SNR of time series, \( m = 7 \) and \( k = 1 \) are fixed. Fig.6 shows the plots of \( E(1) \) versus \( L \) for clean, and noisy signals with SNR = 20dB, 10dB, 6dB and 0dB respectively. For time series of (a) \( x(t) = \sqrt{2} \sin(\pi t) \) and (b) Rössler equation, it is shown that the choice of optimal time delay is insensitive to the SNR of time series. It shows the advantage of the proposed method.

Fig.6 \( E(1) \) versus \( L \) for clean and noisy signals with SNR = 20dB, 10dB, 6dB and 0dB respectively of (a) \( x(t) = \sqrt{2} \sin(\pi t) \) and (b) Rössler equation with \( m = 7 \)

4 Conclusion

Singular entropy (\( SE \)) is applied to choose the optimal time delay in reconstructed dynamics. Algorithmic robustness for the choice of time delay of time series, including the effects of the order of \( SE \) and the corresponding signal to noise rates (SNR), is discussed. It’s found that smaller order of \( SE \) is more preferable, and the choice of time delay is insensitive to the SNR of time series with appropriate order, and it shows the advantages of the proposed method.

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References