Algorithm Robustness Analysis for the Choice of Optimal Time Delay of Phase Space Reconstruction Based on Singular Entropy Method

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Abstract. The method of Delays is commonly used in state space reconstruction of chaotic time series, and the quality of attractor reconstruction of phase space is sensitive to the choice of embedding parameters, i.e. embedding dimension and time delay. In this paper, singular entropy (SE) is applied to choose the optimal time delay in reconstructed dynamical system, and the comparisons of relevant results are made with the correlation function method by using singular value fraction (SVF). The robustness of singular entropy algorithm is further discussed, and it shows the advantage of the proposed method.

Keywords.singular entropy; phase space reconstruction; the optimal time delay; robustness

0 Introduction

The analysis of chaotic time series has become popular in many fields of science and engineering since the work of Packard [1] and Takens [2] who initially introduced the delay-coordinate embedding technique to reconstruct a smooth copy of the underlying dynamics. According to Takens' theorem, a scalar time series $\{x_i\}$, $i=1,2,\dots,N$ can be embedded into a m-dimensional space to form multiple state-space vectors X_i if the embedding dimension m is equal or more than twice the topological n, i.e. $m \ge 2n+1$ .The reconstructed trajectory X is given by

$$X=[X_1,X_2,\cdots,X_{N_m}]^T \qquad (1)$$

The reconstructed state of the system at each discrete time i is

$$X_i=(x_i,x_{i+L},\dots,x_{i+(m-1)L}) \qquad (2)$$

where m is the embedding dimension, L the reconstruction time delay defined as the integer multiple of sampling time τ_s, and N_m the number of vectors in state space which is given as $N_m=N-(m-1)L$.

In fact, the choice of embedding parameters is critical for the quality of attractor reconstruction due to the time series with noise and finite length. An excessively small value of L results in redundancies between successive delay coordinates with little information, while too large value of L may cause successive delay coordinates irrelevant so that the reconstruction attractors are no longer the representative of true dynamics [3].

Different methods have been proposed to choose the value of L based on the criteria of sequential correlation and dimensional expansion. Albano suggested using L as optimal time delay when the autocorrelation function $R_{xx}(L)$ first drops to 1/e of its initial value [4], and Fraser and Swinney proposed to choose the first minimum value of the mutual information function L as optimal time delay instead [5]. The geometrical concepts of *fill factor* [6], *wavering product* [7] and *average displacement* [8] are also applied to measure the expansion of attractors in the state space. Martinerie considered that it's more meaningful to estimate m and L simultaneously with a time window [9], and Kember proposed the recipe of correlation function termed *singular value fraction* (SVF) to choose proper time delay [10]. Such methods as high-order correlation [11], nonlinear correlation functions [12], and singular value spectrum entropy [13] are also proposed to overcome the deficiency of autocorrelation function.

In this paper, on the basis of global *singular value decomposition* (*SVD*), the concept of *singular entropy* (*SE*) [14] is adopted to determine the optimal time delay L. Section 1 presents the definition of singular entropy (*SE*) and its application in the description of information measure. In Section 2, *SE* is applied to choose the optimal time delay L of dynamical reconstruction, where the comparisons of relevant results with SVF method are concerned. Section 3 discusses the influences of the order of SE and the signal to noise rate (*SNR*) of time series on the choice of time delay. The corresponding conclusion is drawn in Section 4.

1 The definition of singular entropy

For a measured discrete time series $\{x_i\}$ that is normalized to have zero mean and unit variance, the SVD of reconstructed trajectory matrix (1) gives

$$X = S\Lambda U^T \qquad (3)$$

where S is the $N_m \times m$ matrix of eigenvectors of XX^T, U the $m \times m$ matrix of eigenvectors of $X^T X$, and Λ $m \times m$ diagonal matrix consisting of singular values σ_i, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$, i.e., $\Lambda = diag(\sigma_1, \sigma_2, ..., \sigma_m)$.

Rearranging equation (3)

$$(XU)^T XU = \Lambda^2 \qquad (4)$$

Thus, σ_i can be considered as a measure of power resident in the column vector of XU whose nonzero amounts represent different patterns of X and information measure of original signal. Λ is also related to the expansion of attractor, where symmetric spread with all equal σ_i is more preferable [10]. Instead of the description of singular value fraction function $f_{sv}(k)$ in references [10] for attractor expansion, in this paper singular entropy (*SE*) is adopted instead and defined as

$$E(k) = \sum_{i=1}^{k} \Delta E_i, \quad i = 1, 2, \cdots, m \qquad (5)$$

where k is the order of singular entropy ($k \leq m$), and ΔE_i the increment of singular entropy in order i which is given by

$$\Delta E_i = -p_i \log p_i \qquad (6)$$

p_i reflects the proportion of the *i*th singular value in the total singular value spectrum that contributes to the global pattern, which is defined as

$$p_i = \sigma_i^2 \Big/ \sum_{i=1}^{m} \sigma_i^2 \qquad (7)$$

Therefore, singular entropy can be considered as information measure of signals [14], and the example is given as follows.

The following of this paper is concerned with the application of SE for the choice of L, and the comparisons with SVF method for dynamical phase reconstruction is made as well.

2 Applications for the choice of L with Singular entropy (*SE*)

The choice of time delay of reconstructed dynamics with singular entropy involves in this section. Three equations of sinusoid and Rössler are employed. k is chosen to be 1 in the examples and whose influence on L will be further discussed in Section 4.

2.1 Choice of L for sinusoid

Fig.1 shows the curve of $E(1)$ versus L

for time series of $x(t)=\sqrt{2}\sin \pi t$ with sample time $\tau_s = 0.01$ s and embedding dimension $m = 2,3,4,5,7,9$ respectively. Larger value of $E(k)$ implies more information from the first k singular values of reconstructed dynamics, which is closely related to preferable expansion of attractors.

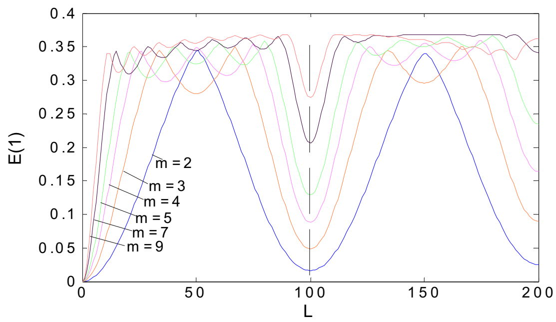


Fig.1 $E(1)$ versus L for a time series of $x(t)=\sqrt{2}\sin \pi t$ with $m = 2,3,4,5,7,9$

In fig.1, the moment when first maximum of $E(1)$ occurs is chosen to represent the optimal time delay denoted as L_m, which is closely related to the time of Δ_M when first minimum $E(1)$ occurs and given by

$$L_m \approx \Delta_M / m \quad (8)$$

Optimal time window Δ_w is defined as

$$\Delta_w = (m-1)L_m \approx (m-1)\Delta_M / m \quad (9)$$

Similar conclusions can be found in reference [10], while the proposed method here applies singular entropy instead of using linear SVF function.

For comparisons, Fig.2 shows the plots of $fsv(1)$ versus L by using the method of SVF for the time series of $x(t)=\sqrt{2}\sin \pi t$ with $m = 2,3,4,5,7,9$ respectively.

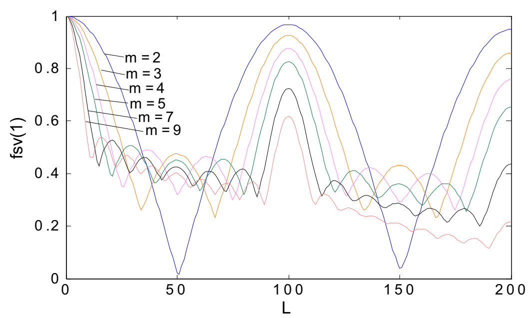


Fig.2 $fsv(1)$ versus L for a time series of $x(t)=\sqrt{2}\sin \pi t$ with $m = 2,3,4,5,7,9$

Table 1 shows the variation of L_m with different m for time series of $x(t)=\sqrt{2}\sin \pi t$, and the results are compared with SVF method as well.

Table 1. the variation of L_m with different m for time series of $x(t)=\sqrt{2}\sin \pi t$ and the comparisons with SVF method.

	L_m		Δ_M	Δ_M / m	Δ_w	m
SE	51	2	51	100	50	
	68	3	34	100	33	
	78	4	26	100	25	
	84	5	21	100	20	
	90	7	15	100	14	
	96	9	12	100	11	
SVF	51	2	51	——	——	
	68	3	34	——	——	
	78	4	26	——	——	
	84	5	21	——	——	
	90	7	15	——	——	
	96	9	12	——	——	

2.2 Choice of L for Rössler equations

Consider Rössler system described by the equations $\dot{x} = -y - z$, $\dot{y} = x + ay$, $\dot{z} = b + z(x-c)$, where $a = 0.15$, $b = 0.20$, $c = 10.0$. The differential equations are solved with numerical integration using a four-order Runge-Kutta algorithm. Initial values are chosen to be $x_0 = 10$, $y_0 = 0$, $z_0 = 0$ and the integral step is fixed at $\tau_s = \pi / 100$. The first 5000 data samples are discarded.

Fig.3 shows the curve of $E(1)$ versus L for Rössler equations using a time series of $x(t)$ with embedding dimension $m = 2,3,4,5,7,9$ respectively.

 DOI: 10.1051/matecconf/20166304027

	L_m	Δ_M	Δ_M / m	Δ_w	m
		48	96		48
48	2				
		32	96		32
64	3				
		24	96		24
72	4				
SE		19	96		19
76	5				
		14	96		14
84	7				
		11	96		11
88	9				
		48	——		——
48	2				
		32	——		——
64	3				
		24	——		——
72	4				
SVF		19	——		——
76	5				
		14	——		——
84	7				
		11	——		——
96	9				

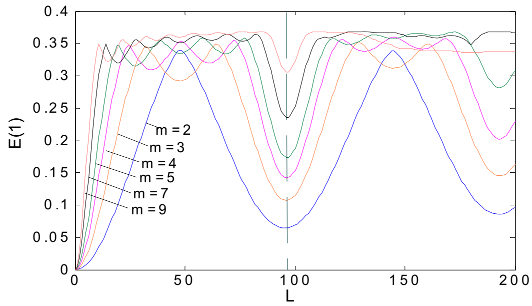


Fig.3 $E(1)$ versus L for Rössler equations using a time series of $x(t)$ with $m = 2,3,4,5,7,9$

Fig.4 shows the curve of $fsv(1)$ versus L for Rössler equations using a time series of $x(t)$ with the same parameters as those of Fig.5.

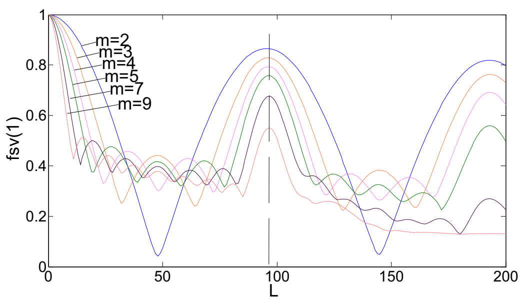


Fig.4 $fsv(1)$ versus L for Rössler equations using a time series of $x(t)$ with $m = 2,3,4,5,7,9$

Table 2 shows the variation of L_m with different m for Rössler equations using a time series of $x(t)$ and resultant comparisons as well.

Table 2 The variation of L_m with different m for time series of Rössler equations and the comparisons with SVF method

3 The robustness analysis of SE method

The robustness of SE method for the choice of L, including the effects of order k and the corresponding signal to noise rates (SNR) of time series, involves in this section.

Fig.5 shows the curves of $E(k)$ versus L for the same time series of (a) $x(t) = \sqrt{2}\sin \pi t$, and (b) Rössler equation respectively as mentioned in Section 2 with fixed $m = 7$ and the variation of k from 1 to 7.

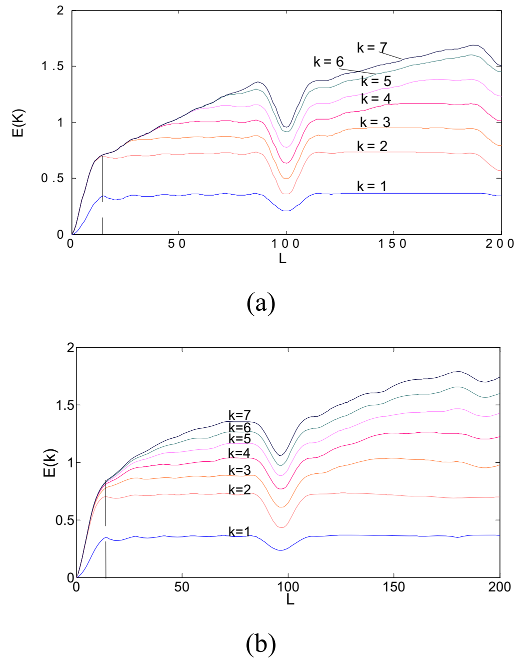
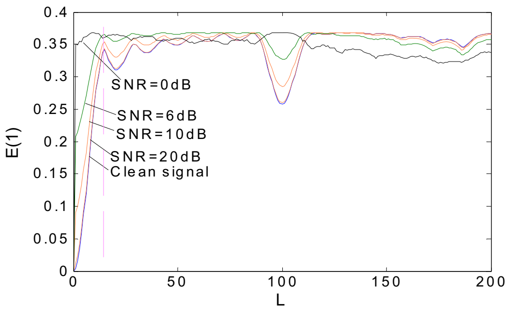


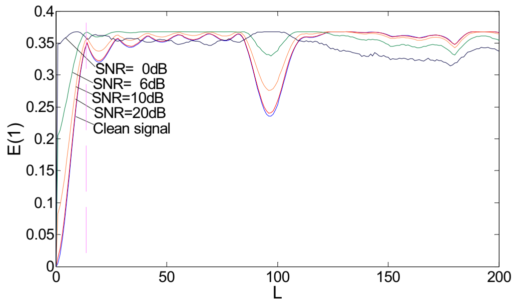
Fig.5 $E(k)$ versus L for time series of (a) $x(t) = \sqrt{2}\sin \pi t$ and (b) Rössler with $m = 7$ and $k = 1,2,3,4,5,6,7$

From Fig .5, the choice of larger k fails to reach satisfactory results, and therefore the smaller one is recommended instead.

To examine the sensitivity of SE method to the SNR of time series, $m = 7$ and $k = 1$ are fixed. Fig.6 shows the plots of $E(1)$ versus L for clean, and noisy signals with SNR=20dB, 10dB, 6dB and 0dB respectively. For time series of (a) $x(t) = \sqrt{2}\sin \pi t$ and (b) Rössler equation, it is shown that the choice of optimal time delay is insensitive to the SNR of time series. It shows the advantage of the proposed method.



(a)



(b)

Fig.6 $E(1)$ versus L for clean and noisy signals with SNR=20dB, 10dB, 6dB and 0dB respectively of (a) $x(t) = \sqrt{2}\sin \pi t$ and (b) Rössler equation with $m = 7$

4 Conclusion

Singular entropy (*SE*) is applied to choose the optimal time delay in reconstructed dynamics. Algorithmic robustness for the choice of time delay of time series, including the effects of the order of *SE* and the corresponding signal to noise rates (*SNR*) ,is discussed. It's found that smaller order of *SE* is more preferable, and the choice of time delay is insensitive to the *SNR* of time series with appropriate order, and it shows the advantages of the proposed method.

Acknowledgements

Part of this work was supported by Natural Science Foundation of China, (Grant NO. 51279107).

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