Study of properties of solutions for quasilinear parabolic systems

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Abstract. In this paper, we concern with degenerate and quasilinear parabolic systems not in divergence form with null Dirichlet boundary conditions and positive initial conditions. The local existence and uniqueness of classical solution are proved. Moreover, It will be proved that all solutions exist globally with homogeneous Dirichlet boundary condition.

1 Introduction

In this paper, we consider the following degenerate and quasilinear parabolic systems not in divergence form:

$$\begin{cases} u_{it} = u_i^{p_i}(\Delta u_i + a_i u_{i+1}), i = 1, 2, \cdots, m, u_{m+1} := u_1 \\ u_i(x, 0) = u_{i0}(x), i = 1, 2, \cdots, m, x \in \Omega, \\ u_i(x, t) = 0, i = 1, 2, \cdots, m, x \in \partial\Omega, t > 0, \end{cases} \quad (1)$$

where a_i , p_i ,are positive constants, $\Omega \in R^N$ is a bounded domain with smooth boundary $\partial\Omega$,The initial data $u_{i0}(x)$, $i = 1, 2, \cdots, m$, satisfy

$$\begin{cases} u_{i0}(x) \in C^1(\bar{\Omega}), u_{i0}(x) > 0, i = 1, 2, \cdots, m, in\ \Omega, \\ u_i(x, t) = 0, \dfrac{\partial u_{i0}}{\partial \eta} < 0, i = 1, 2, \cdots, m, in\ \partial\Omega, \end{cases} \quad (2)$$

here η is the outward normal vector on $\partial\Omega$. The first aim of this paper is to prove the local existence and uniqueness of positive classical solutions. The first aim of this paper is to prove the local existence and uniqueness of positive classical solution of (1) exist globally if and only if $\Pi_{i=1}^m a_i \le \lambda_1^m$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition.

Throughout this paper we say that $(u_1, u_2, \cdots, u_m)$ is a classical solution of a initial and boundary value problem if $(u_1, u_2, \cdots, u_m) \in C^{2,1}(\Omega \times (0,T)) \cap C(\bar{\Omega} \times [0,T))$ for some $T : 0 \le T < +\infty$, and $(u_1, u_2, \cdots, u_m)$ satisfies the differential equations in $\Omega \times (0,T)$ and the initial and boundary conditions continuously of this problem.

This system can be used to describe the development of multiple groups in the dynamics of biological groups where u_i , $i = 1, 2, \cdots, m$ are the densities of the different groups.

Our paper is organized as follows. In Section 2 we prove the local existence, uniqueness and comparison principle of solution to (1). In Section 3 we will prove that the solutions of (1) blows up in finite time if $\Pi_{i=1}^m a_i \ge \lambda_1^m$ and finally we prove that if $\Pi_{i=1}^m a_i \le \lambda_1^m$ then the solution of (1) exists globally.

2 Existence, uniqueness of solutions

In this section, we shall first consider local existence of system (1). Since $u_i, i = 1, 2, \cdots, m$ on the boundary $\partial\Omega$, the equation of (1) is not strictly parabolic type. The standard parabolic theory [1,2] cannot be used directly to prove the local existence of solution to problem (1). To overcome this difficulty, we will use the standard approximate method, see [3], For $\varepsilon > 0$, considering the following approximate problem

$$\begin{cases} u_{i\varepsilon t} = f_{i\varepsilon}(u_{i\varepsilon})(\Delta u_{i\varepsilon} + a_i u_{i+1\varepsilon} \ , i = 1, 2, \cdots, m, x \in \Omega, t > 0, \\ u_{i\varepsilon}(x, 0) = u_{i0}(x) + \varepsilon \ , i = 1, 2, \cdots, m, x \in \Omega, \\ u_{i\varepsilon}(x, t) = \varepsilon \ , i = 1, 2, \cdots, m, x \in \partial\Omega, t > 0 \end{cases} \quad (3)$$

Where $u_{m+1} := u_1$, $f_{i\varepsilon}$, $i = 1, 2, \cdots, m$ are smooth function, $f_{i\varepsilon} \ge (\frac{\varepsilon}{2})^{p_i}$, $i = 1, 2, \cdots, m$

$$f_{i\varepsilon} = \begin{cases} u_i^{p_i} & , u_i \ge \varepsilon \\ (\frac{\varepsilon}{2})^{p_i} & , u_i < \varepsilon \end{cases}$$

The standard parabolic theory shows that (3) admits a unique maximal defined classical solution

$(u_{1\varepsilon},u_{2\varepsilon},\cdots,u_{m\varepsilon})$ which is defined on $[0,T(\varepsilon))$ here . $0<T(\varepsilon)\le\infty$ .The maximum principle implies that $u_{i\varepsilon}\ge\varepsilon$, $i=1,2,\cdots,m$ which gives $f_{i\varepsilon}(u_{i\varepsilon})\ge u_{i\varepsilon}^{pi}$ and hence $(u_{1\varepsilon},u_{2\varepsilon},\cdots,u_{m\varepsilon})$ the following problem

$$\begin{cases} u_{i\varepsilon t}=u_{i\varepsilon}^{pi}(\Delta u_{i\varepsilon}+a_i u_{i+1\varepsilon}),\ x\in\Omega,\ 0<t<T(\varepsilon),\\ u_{i\varepsilon}(x,0)=u_{i0}(x)+\varepsilon,\ \ x\in\Omega,\\ u_{i\varepsilon}(x,t)=\varepsilon,\ \ x\in\partial\Omega,\ 0<t<T(\varepsilon). \end{cases} \tag{4}$$

Where $i=1,2,\cdots,m$, $u_{m+1}\ge u_1$. The fact that $u_{i\varepsilon}\ge\varepsilon$ shows that problem (4) is not degenerate. Since problem (4) is quasimonotone increasing in $(u_{1\varepsilon},u_{2\varepsilon},\cdots,u_{m\varepsilon})$, applying the comparison principle for parabolic systems (.[1, Chapter IV]) we have the following two lemmas.

Lemma 2.1 Assume that positive functions $u_i\in C(\overline{\Omega}\times[0,T(\varepsilon))\cap C^{2,1}(\Omega\times(0,T(\varepsilon)))$ $,i=1,2,\cdots,m$. If $(u_1,u_2,\cdots,u_m)$ is a lower (or upper) solution of (4). Then $(u_1,u_2,\cdots,u_m)\le(\ge)(u_{1\varepsilon},u_{2\varepsilon},\cdots,u_{m\varepsilon})$ on $\overline{\Omega}\times[0,T(\varepsilon)$.

Lemma 2.2 Let $\varepsilon_1<\varepsilon_2$ then $T(\varepsilon_1)<T(\varepsilon_2)$ and $(u_{1\varepsilon_1},u_{2\varepsilon_1},\cdots,u_{m\varepsilon_1})$ $<(u_{1\varepsilon_2},u_{2\varepsilon_2},\cdots,u_{m\varepsilon_2})$ on $\overline{\Omega}\times[0,T(\varepsilon_2)$.

It follows from Lemma 2.2 that there exist $T:0<T\le\infty$ and $(u_1(x,t),u_2(x,t),\cdots u_m(x,t))$ which is defined on $\overline{\Omega}\times[0,T)$ such that $T(\varepsilon)$, $T=\sup_{\varepsilon>o}T(\varepsilon)$ and $(u_{1\varepsilon}(x,t),u_{2\varepsilon}(x,t),\cdots u_{m\varepsilon}(x,t))\to$ $(u_1(x,t),u_2(x,t),\cdots u_m(x,t))$ as $\varepsilon\to 0$.

To discuss the positivity and regularity of $(u_1(x,t),u_2(x,t),\cdots u_m(x,t))$, we should give some estimate of the lower bound of $(u_1(x,t),u_2(x,t),\cdots u_m(x,t))$. Let λ_1 and $\varphi(x)>0(x\in\Omega)$ be respectively the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\varphi=\lambda\varphi \text{ in } \Omega,\quad \varphi=0 \text{ on } \partial\Omega .$$

and think that $\max_{\overline{\Omega}}\varphi(x)=1$. Then $\lambda_1>0$ and $\frac{\partial\varphi}{\partial\eta}<0$ on $\partial\Omega$.

By the assumptions on $(u_{10},u_{20},\cdots u_{m0})$ we see that there exist positive constants k_i $,i=1,2,\cdots,m$ such that

$$\begin{cases} u_{i0}(x)\ge k_i\varphi(x),\ i=1,2,\cdots m,\ x\in\overline{\Omega},\\ \dfrac{\lambda_1}{a_i}\le\dfrac{k_{i+1}}{k_i},\ i=1,2,\cdots m,k_{m+1}:=k_m,\ if\ \prod_{i=1}^m a_i\ge\lambda_1^m,\\ \dfrac{\lambda_1}{a_i}>\dfrac{k_{i+1}}{k_i},\ i=1,2,\cdots m,k_{m+1}:=k_m,\ if\ \prod_{i=1}^m a_i<\lambda_1^m, \end{cases} \tag{5}$$

Lemma 2.3 Let $\varepsilon<1$, $(u_{1\varepsilon},u_{2\varepsilon},\cdots u_{m\varepsilon})$ be the solution of (4), and positive constants $k_1,k_2,\cdots k_m$ satisfy (5). We have the followingestimates:

(i) If $\prod_{i=1}^m a_i\ge\lambda_1^m$, then for $i=1,2,\cdots,m$,

$$u_{i\varepsilon}(x,t)a_i\ge k_i\varphi(x)+\varepsilon,\quad \forall(x,t)\in\overline{\Omega}\times[0,T(\varepsilon)), \tag{6}$$

(ii) If $\prod_{i=1}^m a_i\le\lambda_1^m$, then for $i=1,2,\cdots,m$,

$$u_{i\varepsilon}(x,t)a_i\ge k_i\varphi(x)e^{-\mu t}+\varepsilon,\quad \forall(x,t)\in\overline{\Omega}\times[0,T(\varepsilon)) \tag{7}$$

, where

$$\mu\ge\max\left\{(1+k_i)^{p_i}(\lambda_1-a_i\frac{k_{i+1}}{k_i}),i=1,2,\cdots,m,k_{m+1}:=k_1\right\}$$

.

Proof: (i) When $\prod_{i=1}^m a_i\ge\lambda_1^m$, set $u_{i\varepsilon}=k_i\varphi(x)+\varepsilon,i=1,2,\cdots,m$, In view of (5) , for $u_{m+1}:=u_1$ and, $k_{m+1}:=k_1$ we have,

$$\begin{aligned} &\underline{u}_{i\varepsilon}^{p_i}(\Delta\underline{u}_{i\varepsilon}+a_i\underline{u}_{i+1\varepsilon})=(k_i\varphi+\varepsilon)^{p_i}(-\lambda_1k_i\varphi+a_i(k_{i+1}\varphi+\varepsilon))\\ &>(k_i\varphi+\varepsilon)^{p_i}(a_ik_{i+1}-\lambda_1k_i)\varphi\\ &\ge 0=\underline{u}_{i\varepsilon t},\quad (x,t)\in\Omega\times(0,T(\varepsilon)), \end{aligned}$$

,

$$\underline{u}_{i\varepsilon}(x,t)=\varepsilon,\quad (x,t)\in\partial\Omega\times(0,T(\varepsilon))$$

$$\underline{u}_{i\varepsilon}(x,0)\le\underline{u}_{i\varepsilon}(x,0),\quad x\in\overline{\Omega}.$$

This shows that $(\underline{u}_{1\varepsilon},\underline{u}_{2\varepsilon},\cdots\underline{u}_{m\varepsilon})$ is a lower solution of (4). Lemma 2.1 implies that (6) holds.

When $\prod_{i=1}^m a_i<\lambda_1^m$, set $\underline{u}_{i\varepsilon}(x,t)=k_i\varphi(x)e^{-\mu t}+\varepsilon,i=1,2,\cdots,m$ Then from (5) and (7), for $u_{m+1}:=u_1$ and $k_{m+1}:=k_1$ we have,

$$\underline{u}_{i\varepsilon t}=-\mu k_i\varphi e^{-\mu t},$$

$$\begin{aligned} &\underline{u}_{i\varepsilon}^{p_i}(\Delta u_{i\varepsilon}+a_i\underline{u}_{i+1\varepsilon})\\ &=(k_i\varphi e^{-\mu t}+\varepsilon)^{p_i}(-k_i\lambda_1\varphi e^{-\mu t}+a_ik_{i+1}\varphi e^{-\mu t}+a_i\varepsilon)\\ &>(k_i\varphi e^{-\mu t}+\varepsilon)^{p_i}(a_ik_{i+1}-k_i\lambda_1)\varphi e^{-\mu t}\\ &\ge(\varepsilon+k_i)^{p_i}(a_ik_{i+1}-k_i\lambda_1)\varphi e^{-\mu t}\\ &>(1+k_i)^{p_i}(a_ik_{i+1}-k_i\lambda_1)\varphi e^{-\mu t}\\ &\ge-\mu k_i\varphi e^{-\mu t}=\underline{u}_{i\varepsilon t},\quad (x,t)\in\Omega\times(0,T(\Omega)) \end{aligned}$$

$\underline{u}_{i\varepsilon}(x,t)=\varepsilon, \quad (x,t)\in\partial\Omega\times(0,T(\varepsilon))$,

$\underline{u}_{i\varepsilon}(x,0)\le u_{i\varepsilon}(x,0), \quad x\in\overline{\Omega}$.

The same reason as that of (i) shows that (6) holds. The proof is completed.

Theorem 2.1 Problem (1) has a positive classical solution $(u_1,u_2,\cdots u_m)\in[C^{2+\beta,1+\beta/2}(\Omega\times(0,T))\cap C(\overline{\Omega}\times[0,T))]^m$ for some $\beta:0<\beta<1$, and $T:0<T\le\infty$.

Proof: Lemma 2.3 shows that $(u_1,u_2,\cdots u_m)$, the limit of $(u_{1\varepsilon},u_{2\varepsilon},\cdots u_{m\varepsilon})$, is positive in $\Omega\times(0,T)$. In view of Lemma 2.2 and Lemma 2.3, the classical results of local Schauder estimates (see[4]) imply that $u_{i\varepsilon}\in C^{2+\alpha,(1+\alpha)/2}(\Omega\times(0,T(\varepsilon))$ for some $\alpha>0$. Furthermore, for any $\Omega_0\subset\Omega$ and $0<\tau<T_0<T$, we have

$(u_{1\varepsilon},u_{2\varepsilon},\cdots u_{m\varepsilon})\to(u_1,u_2,\cdots u_m)$ in $[C^{2+\beta,1+\beta/2}(\overline{\Omega}_0\times[\tau,T_0])]^m$, $0<\beta<\alpha$, $\varepsilon\to0^+$.

Therefore, $(u_1,u_2,\cdots u_m)$ belongs to $C^{2+\beta,1+\beta/2}(\Omega\times(0,T(\varepsilon))$ and satisfies the differential equations of (1) in $\Omega\times(0,T)$. Fix $\varepsilon_0:0<\varepsilon_0<1$. For any $\Omega_0\subset\Omega$ and $0<\varepsilon<\varepsilon_0$, From Lemma 2.1, we have $\underline{u}_{i\varepsilon}(x,t)<u_{i\varepsilon_0}(x,t), i=1,2,\cdots,m$ on $\overline{\Omega}_0\times[0,T(\varepsilon_0)/2]$, in view of Lemma 2.3, the L^p theory and embedding theorem show that the $C^{\alpha,\alpha/2}(\overline{\Omega}_0\times[0,T(\varepsilon_0)/2])$ norm of $u_{i\varepsilon}, i=1,2,\cdots,m$ is uniformly

bounded above for all $\varepsilon<\varepsilon_0$. Hence $(u_{1\varepsilon},u_{2\varepsilon},\cdots u_{m\varepsilon})\to(u_1,u_2,\cdots u_m)$ in $[C^{\beta,\beta/2}(\overline{\Omega}_0\times[0,T(\varepsilon_0)/2])]^m$, $0<\beta<\alpha$, $\varepsilon\to0^+$, which implies that $u_i\in C(\Omega\times[0,T]), i-1,2,\cdots,m$. Similar to the arguments of [4,5,6] we can prove that $(u_1,u_2,\cdots u_m)$ is continuous on $\partial\Omega\times(0,T)$. Using the initial and boundary conditions of (4) we see that $(u_1,u_2,\cdots u_m)$ satisfies the initial and boundary conditions of (1), i.e. $(u_1,u_2,\cdots u_m)\in[C^{2+\beta,1+\beta/2}(\Omega\times(0,T)\cap C[0,T))]^m$ is a classical solution of (1). The proof is completed.

Next, we will prove the uniqueness of positive classical solution of (1).

Theorem 2.2 Problem (1) has a unique positive classical solution $(u_1,u_2,\cdots u_m)\in[C^{2+\beta,1+\beta/2}(\Omega\times(0,T)\cap C(\overline{\Omega\times[0,T)})]^m$, for some $\beta:0<\beta<1$ and $T:0<T\le\infty$. Moreover, if $T<\infty$ then $\overline{\lim_{t\to T^-}}\max_{\Omega}u(\cdot,t)=\infty$

Proof: Suppose that $(v_1,v_2,\cdots v_m)$ is also a positive classical solution of (1). By Lemma 2.1 we have that $v_i<u_{i\varepsilon}, i=1,2,\cdots,m$ and hence $v_i<u_i, i=1,2,\cdots,m$. For $\delta>0$ sufficiently small, denote $\Omega_\delta=\{x\in\Omega\,|\,dist(x,\partial\Omega)\ge\delta\}$. Let λ_δ and $\varphi_\delta(x)>0$ be respectively the first eigenvalue and the corresponding eigenfunction of $-\Delta$ in Ω_δ with homogeneous boundary condition, and think that $\max_{\Omega_\delta}\varphi_\delta=1$. Multiplying the equations of (1) $\varphi_\delta/u_i^{p_i}, i=1,2,\cdots,m$ respectively, and integrating the results by parts we have

$$\int_{\Omega_\delta}\xi_i(u_i)\varphi_\delta dx=\int_{\Omega_\delta}\xi_i(u_{i0})\varphi_\delta dx-\int_0^t\int_{\partial\Omega_\delta}u_i\frac{\partial\varphi_\delta}{\partial\eta}dS_xdt$$
$$+a_i\int_0^t\int_{\Omega_\delta}u_{i+1}\varphi_\delta dxdt-\lambda_\delta\int_0^t\int_{\Omega_\delta}u_i\varphi_\delta dxdt, \tag{8}$$

where $\xi_i(u_i)=\begin{cases}\dfrac{u_i^{1-p_i}}{1-p_i}, if\ p_i\ne1,\\ Inu_i, if\ p_i=1,\end{cases}$ $i=1,2,\cdots,m$, $u_{m+1}:=u_1$.

Similarly

$$\int_{\Omega_\delta}\xi_i(v_i)\varphi_\delta dx=\int_{\Omega_\delta}\xi_i(u_{i0})\varphi_\delta dx-\int_0^t\int_{\partial\Omega_\delta}v_i\frac{\partial\varphi_\delta}{\partial\eta}dS_xdt$$
$$+a_i\int_0^t\int_{\Omega_\delta}v_{i+1}\varphi_\delta dxdt-\lambda_\delta\int_0^t\int_{\Omega_\delta}v_i\varphi_\delta dxdt, \tag{9}$$

where $i=1,2,\cdots,m, v_{m+1}:=v_1$. Then (8) subtract (9) respectively, we have

$$\int_{\Omega_\delta}[\xi_i(u_i)-\xi(v_i)]\varphi_\delta dx=-\int_0^t\int_{\partial\Omega_\delta}(u_i-v_i)\frac{\partial\varphi_\delta}{\partial\eta}dS_xdt$$
$$+a_i\int_0^t\int_{\Omega_\delta}(u_{i+1}-v_{i+1})\varphi_\delta dxdt$$
$$-\lambda_\delta\int_0^t\int_{\Omega_\delta}(u_i-v_i)\varphi_\delta dxdt. \tag{10}$$

For any given $T_0:0<T_0<T$, denot $M=\max_{\overline{\Omega}\times[0,T_0]}(\sum_{i=1}^m u_i)$. Applying $v_i<u_i, i=1,2,\cdots,m$, we get that

$$\int_{\Omega_\delta}[\xi_i(u_i)-\xi_i(v_i)]\varphi_\delta dx$$
$$=\int_{\Omega_\delta}(\int_0^1\xi_i'(v_i+s(u_i-v_i))ds)(u_i-v_i)\varphi_\delta dx$$

$$= \int_{\Omega_\delta} \left(\int_0^1 (v_i + s(u_i - v_i))^{-p_i} ds\right)(u_i - v_i)\varphi_\delta dx$$

$$\geq M^{-p_i} \int_{\Omega_\delta} (u_i - v_i)\varphi_\delta dx . \qquad (11)$$

From (10) and (11) it follows that

$$\int_{\Omega_\delta} [\sum_{i=1}^{m} (u_i - v_i)]\varphi_\delta dx$$

$$\leq -M_1 \int_0^t \int_{\partial\Omega_\delta} \sum_{i=1}^{m} (u_i - v_i) \frac{\partial \varphi_\delta}{\partial \eta} dS_x dt$$

$$+M_2 \int_{\Omega_\delta} [\sum_{i=1}^{m} (u_i - v_i)]\varphi_\delta dxdt ,$$

where $M_1 = \max\left\{M^{p_i}, i = 1, 2, \cdots, m\right\}$, $M_2 = \sum_{i=1}^{m} M^{p_i}(a_i + \lambda_\delta)$. Applying Gronwall's lemma and then taking the limit as $\delta \to 0$ it follows that $u_i \equiv v_i, i = 1, 2, \cdots, m$. The proof is completed.

Finally we will give the comparison principle without proof for Problem (1).

Theorem 2.3 Assume that $(\overline{u}_{10}(x), \overline{u}_{20}(x), \cdots, \overline{u}_{m0}(x))$ and $(\underline{u}_{10}(x), \underline{u}_{20}(x), \cdots, \underline{u}_{m0}(x))$ are continuous positive functions in Ω and satisfy (2). Let $(\overline{u}_1(x), \overline{u}_2(x), \cdots, \overline{u}_m(x))$ and $(\underline{u}_1(x), \underline{u}_2(x), \cdots, \underline{u}_m(x))$ be the positivesolutions of (1) with initial data $(\overline{u}_{10}(x), \overline{u}_{20}(x), \cdots, \overline{u}_{m0}(x))$ and $(\underline{u}_{10}(x), \underline{u}_{20}(x), \cdots, \underline{u}_{m0}(x))$, respectively. Denote by $\overline{T}$ and $\underline{T}$ their maximal existence time, If $(\overline{u}_{10}(x), \overline{u}_{20}(x), \cdots, \overline{u}_{m0}(x)) \geq (\underline{u}_{10}(x), \underline{u}_{20}(x), \cdots, \underline{u}_{m0}(x))$ in Ω, then $\overline{T} \leq \underline{T}$ and

$$(\overline{u}_1(x), \overline{u}_2(x), \cdots, \overline{u}_m(x)) \geq (\underline{u}_1(x), \underline{u}_2(x), \cdots, \underline{u}_m(x)),$$
$$\forall (x,t) \in \overline{\Omega} \times [0, \overline{T}) .$$

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