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Numerical Solutions for Convection-Diffusion Equation through Non-Polynomial Spline

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Abstract. In this paper, numerical solutions for convection-diffusion equation via non-polynomial splines are studied. We purpose an implicit method based on non-polynomial spline functions for solving the convection-diffusion equation. The method is proven to be unconditionally stable by using Von Neumann technique. Numerical results are illustrated to demonstrate the efficiency and stability of the purposed method.

1 Introduction

In this paper, we present an implicit non-polynomial spline functions based scheme for the numerical solution of the following one dimensional convection-diffusion equation,

$$u_t + \varepsilon u_x + \gamma u_{xx} = g(x,t),\ 0 \le x \le 1,\ t \ge 0 \quad (1)$$

subject to the boundary conditions

$$u(0,t) = g_0(t),\ u(1,t) = g_1(t),\ t \ge 0 \quad (2)$$

and with the initial condition

$$u(x,0) = f(x),\ 0 \le x \le 1 \quad (3)$$

where $\varepsilon > 0$, is the phase speed and $\gamma < 0$, is the viscosity coefficient. In natural systems, diffusion is one of the important mechanism which means that the mean square displacement of a diffusion particle is in a linear relation with time. The convection-diffusion equation describes the various physical phenomena where energy is transformed inside a physical system due to the combination of convection as well as diffusion processes. It arises in various fields of applied sciences and engineering such as oil reservoir simulations, transport of mass and energy, global weather production, dispersion of diffusion process, due to these vide variety of physical implementations a great deal of researches have been done for the numerical and closed form solutions of convection diffusion type equation for eg., Jain and Aziz [4] used the adaptive spline function approximation for the numerical solution of convection-diffusion equation. Mohammadi [5] applied exponential B-spline collocation method for solving convection-diffusion equation with Dirichlet's type boundary conditions. Caglar et. al. [6] presented B-spline solutions for a convection-diffusion equation. Ravi Kanth and Aruna [1] implemented the differential transform method for solving linear and nonlinear Klein-Gordon equation. Mohammadi [2] presented the exponential spline approach and Lin [3] constructed the parametric cubic spline method for the numerical solution of non-linear Schrodinger equation.

2 Derivation of the Numerical Scheme

For the positive integers N and M, let the partition of $[a,b]\times[0,T]$ be defined by $\Omega : \Omega_h \times \Omega_k$ where $\Omega_h = \{x_i \mid x_i = a + ih, 0 \le i \le N\}$,

$\Omega_k = \{t_j \mid t_j = jk, 0 \le j \le M\}$ and $k = \dfrac{T}{M}$, $h = \dfrac{b-a}{N}$ are temporal and spatial step size respectively.

Let $Z_i^j = Z(x_i, t_j)$ be an approximation to $u_i^j = u(x_i, t_j)$, obtained by the segment $P_i(x_i, t_j)$ of the mixed spline function passing through the points (x_i, Z_i^j) and (x_{i+1}, Z_{i+1}^j). Each segment can be written as [7].

$$\begin{aligned} P_i(x,t_j) &= a_i(t_j)\cos w(x-x_i) + b_i(t_j)\sin w(x-x_i) \\ &+ c_i(t_j)(x-x_i) + d_i(t_j) \end{aligned} \quad (4)$$

for each $i = 0,1,2,\cdots,N-1$. To obtain the coeffcients $a_i(t_j), b_i(t_j), c_i(t_j)$ and $d_i(t_j)$ in terms of Z_i^j, Z_{i+1}^j, S_i^j and S_{i+1}^j, let us define

$$\begin{aligned} &P_i(x_i,t_j) = Z_i^j,\ P_i(x_{i+1},t_j) = Z_{i+1}^j, \\ &P_i^{(2)}(x_i,t_j) = S_i^j,\ P_i^{(2)}(x_{i+1},t_j) = S_{i+1}^j \end{aligned} \quad (5)$$

where

$$P_i^{(2)}(x,t) = \frac{\partial^2}{\partial x^2} P_i(x,t)$$

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By using Eqs. (4) and (5), we obtain by a straightforward calculation,

$$a_i = -\frac{h^2}{\theta^2} S_i^j,\ b_i = \frac{h^2\left(\cos\theta S_i^j - S_{i+1}^j\right)}{\theta^2 \sin\theta}$$

$$c_i = \frac{\left(Z_{i+1}^j - Z_i^j\right)}{h} + \frac{h\left(S_{i+1}^j - S_i^j\right)}{\theta^2}$$

$$d_i = \frac{h^2}{\theta^2} S_i^j + Z_i^j \qquad (6)$$

where
$a_i = a_i(t_j),\ b_i = b_i(t_j),\ c_i = c_i(t_j),\ d_i = d_i(t_j)$ and $\theta = hw$.

Using the following continuity condition of first derivative at point $x = x_i$, i.e.,

$$P_i^{(1)}(x_i, t_j) = P_{i-1}^{(1)}(x_i, t_j)$$

The equations (4) and (6) yield the relation

$$\frac{1}{h^2}\left[Z_{i-1}^j - 2Z_i^j + Z_{i+1}^j\right] = \alpha S_{i-1}^j + \beta S_i^j + \alpha S_{i+1}^j \qquad (7)$$
$$i = 1, 2, \cdots N-1$$

where $\alpha = \frac{1}{\sin\theta} - \frac{1}{\theta^2},\ \beta = -\frac{2\cos\theta}{\theta\sin\theta} + \frac{1}{\theta^2}$

As $w \to 0$, Eqn. (7) transformed into ordinary cubic spline relation as in [7].

Considering the equation (7) at j^{th} and $(j+1)^{th}$ time levels, so that we can get,

$$\frac{1}{h^2}\left[\left(\frac{Z_{i-1}^{j+1} + Z_{i-1}^j}{2}\right) - \left(Z_i^{j+1} + Z_i^j\right) + \left(\frac{Z_{i+1}^{j+1} + Z_{i+1}^j}{2}\right)\right] =$$
$$\alpha\left(\frac{S_{i-1}^{j+1} + S_{i-1}^j}{2}\right) + \beta\left(\frac{S_i^{j+1} + S_i^j}{2}\right) + \alpha\left(\frac{S_{i+1}^{j+1} + S_{i+1}^j}{2}\right) \qquad (8)$$
$$, i = 1, 2, 3, \ldots, N-1$$

Remark :
The truncation error for Eq. (8) is

$$T_i^j = \left(u_{i-1}^{j+1} + u_{i-1}^j\right) - 2\left(u_i^{j+1} + u_i^j\right) + \left(u_{i+1}^{j+1} + u_{i+1}^j\right)$$
$$-\alpha^* D_x^2\left(u_{i-1}^{j+1} + u_{i-1}^j\right) - \beta^* D_x^2\left(u_i^{j+1} + u_i^j\right)$$
$$-\alpha^* D_x^2\left(u_{i+1}^{j+1} + u_{i+1}^j\right)$$

Where $\alpha^* = \alpha h^2$ and $\beta^* = \beta h^2$. Now expanding the above Eqn, in terms $u(x_i, t_j)$ by using Taylor series, we obtain

$$T_i^j = \left(2h^2 - 2(2\alpha^* + \beta^*)\right) D_x^2 u_i^j + h^2\left(\frac{1}{6} - 2\alpha^*\right) D_x^4 u_i^j$$
$$+ h^4\left(\frac{1}{180} - \frac{\alpha^*}{6}\right) D_x^6 u_i^j + \ldots$$

We can observe from the above expression that if $2\alpha^* + \beta^* = h^2$ then our scheme is of $O(h^2)$ and if $2\alpha^* + \beta^* = h^2$ with $\alpha^* = \frac{h^2}{12}$, then our scheme is of $O(h^4)$.

At the grid point (x_i, t_j) equation (1) can be rewritten in discretized form as

$$S_i^j = \left(Z_{xx}\right)_i^j = \frac{1}{\gamma}\left[g_i^j - \left(Z_t\right)_i^j - \grave{o}\left(Z_x\right)_i^j\right] \qquad (9)$$

To obtain the approximation for Eq. (9), based on Taylor series using the following finite difference approximations

$$\left(Z_x\right)_i^j = \frac{Z_{i+1}^j - Z_{i-1}^j}{2h} + O\left(h^2\right)$$

$$\left(Z_x\right)_{i-1}^j = \frac{-Z_{i+1}^j + 4Z_i^j - 3Z_{i-1}^j}{2h} + O\left(h^2\right)$$

$$\left(Z_x\right)_{i+1}^j = \frac{3Z_{i+1}^j - 4Z_i^j + Z_{i-1}^j}{2h} + O\left(h^2\right)$$

Applying the Crank-Nicolson scheme to Eq. (9) along with the above finite difference approximations, we obtain,

$$\frac{\left(S_i^{j+1} + S_i^j\right)}{2} = \frac{g_i^{j+1} + g_i^j}{2\gamma} - \frac{\left(Z_i^{j+1} - Z_i^j\right)}{k\gamma}$$
$$-\frac{\varepsilon}{4h\gamma}\left(Z_{i+1}^{j+1} - Z_{i-1}^{j+1} + Z_{i+1}^j - Z_{i-1}^j\right)$$
$$, i = 1, 2, \cdots N-1, \text{ and } j \geq 0 \qquad (10)$$

$$\frac{\left(s_{i-1}^{j+1} + s_{i-1}^j\right)}{2} = \frac{\left(g_{i-1}^{j+1} + g_{i-1}^j\right)}{2\gamma} - \frac{\left(Z_{i-1}^{j+1} + Z_{i-1}^j\right)}{k\gamma}$$
$$-\frac{\varepsilon}{4h\gamma}\left(-Z_{i+1}^{j+1} + 4Z_i^{j+1} - 3Z_{i-1}^{j+1} - Z_{i+1}^j + 4Z_i^j - 3Z_{i-1}^j\right),$$
$$i = 1, 2, 3, ..., N-1, and\ j \geq 0 \qquad (11)$$

$$\frac{\left(s_{i+1}^{j+1} + s_{i+1}^j\right)}{2} = \frac{\left(g_{i+1}^{j+1} + g_{i+1}^j\right)}{2\gamma} - \frac{\left(Z_{i+1}^{j+1} + Z_{i+1}^j\right)}{k\gamma}$$
$$-\frac{\varepsilon}{4h\gamma}\left(3Z_{i+1}^{j+1} - 4Z_i^{j+1} + Z_{i-1}^{j+1} + 3Z_{i+1}^j - 4Z_i^j + Z_{i-1}^j\right),$$
$$i = 1, 2, 3, ..., N-1, \text{ and } j \geq 0 \qquad (12)$$

Using the Eqs.(10-12) in Eq.(8), we obtain,

$$\left(2\gamma k-\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)Z_{i-1}^{j+1}+\left(-4\gamma k+4h^2\beta\right)Z_i^{j+1}$$
$$+\left(2\gamma k+\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)Z_{i+1}^{j+1}$$
$$=\left(-2\gamma k+\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)Z_{i-1}^{j}+\left(4\gamma k+4h^2\beta\right)Z_i^{j}$$
$$+\left(-2\gamma k-\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)Z_{i+1}^{j}$$
$$+2\left(\alpha h^2kg_{i-1}^{j+1}+\beta h^2kg_i^{j+1}+\alpha h^2kg_{i+1}^{j+1}\right)$$
$$+2\left(\alpha h^2kg_{i-1}^{j}+\beta h^2kg_i^{j}+\alpha h^2kg_{i+1}^{j}\right)$$
$$i=1,2,3,\ldots,N-1,\text{and } j\geq 0 \qquad (13)$$

The system (13) contains $(N+1)$ equations with $(N+1)$ unknowns. To get a solution to this system, we need two additional equations. These equations are obtained from the boundary conditions (2), can be written in the discretized form as

$$Z_0^j=g_0\left(t_j\right),\ Z_N^j=g_1\left(t_j\right),\ j\geq 0$$

3 Stability analysis

The stability has been proven through the Von Neumann technique,the numerical solution can be expressed by means of a Fourier series

$$Z_i^j=\xi^j\exp\left(I\varphi ih\right) \qquad (14)$$

Where $I=\sqrt{-1},\phi$ is the wave number and ξ^j is the amplitude at time level j. Substituting Eqn.(14) in Eqn.(13), we obtained

$$\xi^{j+1}\left\{\begin{array}{l}\left(2\gamma k-\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)\exp\left(I\left(i-1\right)\sigma\right)\\+\left(-4\gamma k+4h^2\beta\right)\exp\left(Ii\sigma\right)+\left(2\gamma k+\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)\exp\left(I\left(i+1\right)\sigma\right)\end{array}\right\}$$
$$=$$
$$\xi^{j}\left\{\begin{array}{l}\left(-2\gamma k+\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)\exp\left(I\left(i-1\right)\sigma\right)\\+\left(4\gamma k+4h^2\beta\right)\exp\left(Ii\sigma\right)+\left(-2\gamma k-\varepsilon kh\left(2\alpha+\beta\right)+4h^2\alpha\right)\exp\left(I\left(i+1\right)\sigma\right)\end{array}\right\} \qquad (15)$$

where $\sigma=\phi h$. After some manipulations in Eqn. (15) and then using the Euler's formula, we get

$$\xi=\frac{X_1+IY_1}{X_2+IY_2}$$

Where,

$$X_1=4\gamma k\left(1-\cos\sigma\right)+4h^2\left(2\alpha\cos\sigma+\beta\right),$$
$$X_2=-4\gamma k\left(1-\cos\sigma\right)+4h^2\left(2\alpha\cos\sigma+\beta\right)$$
$$Y_1=-Y_2=-2\varepsilon kh\left(2\alpha+\beta\right)\sin\sigma$$

For stability as the time increases, we want the amplification factor ξ^j must satisfy $|\xi|\leq 1$ and for this we must have, $A=X_1^2+Y_1^2-X_2^2-Y_2^2\leq 0$

and since,

$$A=4\gamma kh^2\left(1-\cos\sigma\right)\left(2\alpha\cos\sigma+\beta\right)$$

As $A\leq 0$, provided $\alpha>0,\ \beta>0$ such that $\beta\geq 2\alpha$, so $|\xi|\leq 1$, thus the proposed method is unconditionally stable if $\beta\geq 2\alpha$.

4 Numerical results

In this section, in order to demonstrate the accuracy and effectiveness of the proposed scheme a few numerical evidences are given,

Example 1. Consider the one dimensional convection diffusion equation,

$$u_t-u_{xx}=g(x,t),\ 0\leq x\leq 1,\ 0\leq t\leq 1$$

subject to the boundary conditions

$$u(0,t)=0,\ u(1,t)=0,\ 0\leq t\leq 1$$

and with the initial condition

$$u(x,0)=0,\ 0\leq x\leq 1$$

where, $g(x,t)=2t\sin(2\pi x)+4\pi^2t^2\sin(2\pi x)$. The exact solution of this problem is [8]

$$u(x,t)=t^2\sin(2\pi x)$$

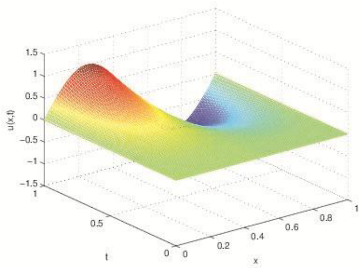

Figure 1. The numerical solution for Example 1 with $h=0.01$, $k=0.01$, $\alpha=1/12$ and $\beta=10/12$.

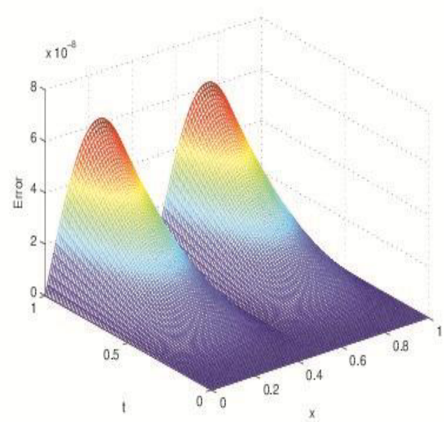

Figure 2. Absolute errors for Example 1 with $h = 0.01$, $k = 0.01$, $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$.

Example 2. Consider the one dimensional convection diffusion equation,

$$u_t + \varepsilon u_x + \gamma u_{xx} = g(x,t),\ 0 \le x \le 1,\ 0 \le t \le 1$$

subject to the boundary conditions

$$u(0,t) = e^{qt},\ u(1,t) = e^{p+qt},\ 0 \le t \le 1$$

and with the initial condition

$$u(x,0) = e^{px},\ \ 0 \le x \le 1$$

where, $g(x,t) = 0$. The exact solution of this problem is

$u(x,t) = e^{px+qt}$

and the parameters $\varepsilon = 0.1,\ \gamma = -0.02,$

$p = 1.17712434446770$ and $q = -0.09$.

The L_∞ and L_2 errors along with the order of convergence are reported in Table 1, which shows that the purposed method is having fourth order convergency for $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$. Table 2, represents the absolute errors for Example 2 at different spatial and time levels. From this table we can observe that the numerical results are in excellent agreement with the analytical solutions. Figure 1 and 3 exhibits the numerical solutions for Examples 1 and Example 2, respectively. It is noticed that our numerical results are in good accordance with the analytic solutions. Figure 2 and 4 shows the absolute errors for Examples 1 and Example 2 respectively, and it can be observe from these figures that our method supports the theoretical results.

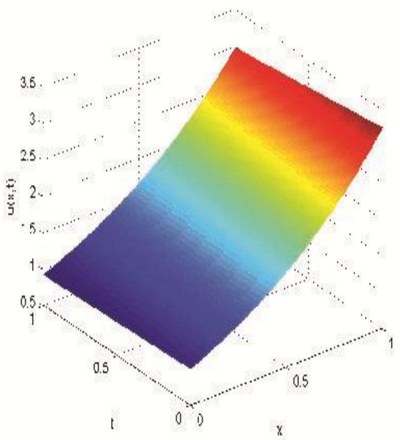

Figure 3. The numerical solution for Example 2 with $h = 0.01$, $k = 0.001$, $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$.

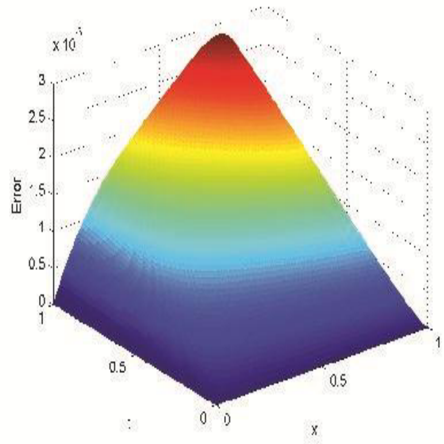

Figure 4. Absolute errors for Example 1 with $h = 0.01$, $k = 0.001$, $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$.

5 Conclusion

In this paper, we have studied the numerical solutions for the convection-diffusion type equation via non-polynomial spline functions. The method is shown unconditionally stable by using Von Neumann stability process. Numerical results shows that the purposed method results are in excellent agreement with the analytical solutions. Illustrative examples are given to demonstrate the applicability and accuracy of the proposed method.

Table 1. Convergence rates for Example 1 with $h = 0.01,\ k = 0.01,\ \alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$.

h	L_∞	**Order**	L_2	**Order**
$\frac{1}{4}$	2.6659e-2	-	1.8850e-2	-
$\frac{1}{8}$	1.5447e-3	4.1092	1.0922e-3	4.1092
$\frac{1}{16}$	9.4775e-5	4.0266	6.7016e-5	4.0266
$\frac{1}{32}$	5.8963e-6	4.0066	4.1693e-6	4.0066
$\frac{1}{64}$	3.6809e-7	4.0016	2.6028e-7	4.0016

Table 2. Absolute errors for Example 2
with $h = 0.01$, $k = 0.01$, $\alpha = 1/12$ and $\beta = 10/12$.

(x,t)	**Errors**
(0.1,0.1)	1.4577e-7
(0.2,0.2)	3.3687e-7
(0.3,0.3)	5.6662e-7
(0.4,0.4)	8.4327e-7
(0.5,0.5)	1.1754e-6
(0.6,0.6)	1.5710e-6
(0.7,0.7)	2.0172e-6
(0.8,0.8)	2.3782e-6
(0.9,0.9)	2.1371e-6

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