MATEC Web of Conferences 47, 02016 (2016)
DOI: 10.1051/matecconf/20164702016

An Orthotropic Membrane Model for the Large Deformation Analysis and Snapping Phenomena of the Dome Inflated

Shohei Yamada[1,a], Katsushi Ijima[1], Hiroyuki Obiya[1] and Muhammad Nizam Zakaria[2]

[1]*Department of Civil Engineering and Architecture, Graduate School of Science and Engineering, Saga University, Honjo, Saga 840-8502, Japan*
[2]*Department of Structural and Material Engineering, Faculty of Civil and Environmental Engineering, University Tun Hussein Onn Malaysia, 86400 Parit Raja, Johor, Malaysia*

Abstract. The paper proposes a method to enable analyzing any large deformation of membrane structures. If the analysis uses even very small rigidity against compression in the structures, the computation becomes multi-bifurcated problem and unstable. The original compression-free model used in the method keeps the structures in tension field and this makes computing the deformation always very stable. The paper uses the advantage of the method and shows a snapping phenomenon during constructing a membrane dome. Though the snapping phenomenon is not so public, the phenomenon occasionally happens in the construction site and it causes the destruction.

1 Introduction

Since membrane structures are light, easily expandable and compactly folded, the structures are widely used in various ways. The mechanical characteristic of the membrane structure is that tension in the membrane keeps the structural shape and that the structure in small tension is so flexible that is easily deformed. Therefore, when the structure is constructed or expanded, the characteristic brings unexpected phenomena, such as snapping of a membrane dome shown in the paper. In order to estimate exactly those phenomena, we need a method that can stably compute any state of the membrane structure.

The authors are proposing a method that uses the compression-free unit composed by cables [1], [2]. The unit in tension has almost the same stiffness as that of the triangular element with uniform strain in it. If compressive force in the element occurs even very small, computing the equilibrium necessarily becomes a multi-bifurcated problem. Since the compressive force in the membrane is essentially very small, using the compression-free unit is appropriate and besides the use makes the analysis completely free from the multi-bifurcated problem.

However, the compression-free assumption occasionally brings unpractical solutions as a result from setting a certain boundary condition of the membrane structure. The unpractical solution is, for example that the area of a membrane element becomes to end zero. The modification to the fault is settled with taking into considering bending moment in a side of a triangular element. Using the end shear forces balanced with the end moment in the element keeps only the translational degrees of freedom at each node without increasing a degree of nodal freedom. The idea has been already

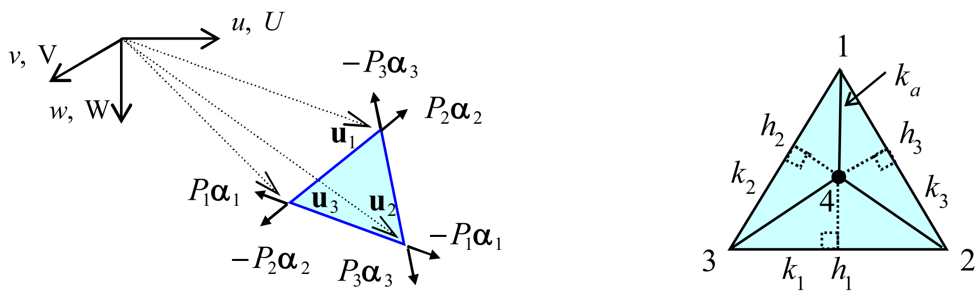
[a] Corresponding author : 15577018@edu.cc.saga-u.ac.jp

published by many studies on analyzing shell structures [3-5]. Membrane material generally has so small rigidity against the bending that the end moment in the element side is very small. However, the end shear force derived from the small end moment can resist shrinking the element into ending zero area.

The method can stably compute equilibrium solutions in any state of membrane structures. An experiment of large deformation of a suspended membrane has verified the validity of the computation by using the method [6]. As an example of analyzing the large deformation, the paper shows snapping phenomena of a membrane dome occurring in the inflation. There was an actual accident that the phenomenon extended to a tense state before the occurrence of the snapping during inflating a large-scale membrane dome. If the dome came to just before the snapping, the dome surly ended broken.

2 Concept of the Theory

The method in the paper uses different procedure from that of the popular FEM. The strain energy of an element is defined by the positions of the nodes connected to the apexes of the element. In the method, the energy principal is applied to each element, and it gives the element end forces. The end forces in this paper are classified into the forces in the plane of the compression-free element and the end shear forces in the normal direction to the element plane. The method finds the nodal positions that satisfy the equilibrium equation of those element end forces and the nodal forces working at each node.



(a) End forces and the position vectors in the element

(b) Compression-free unit composed by 6 axial members

Figure 1. Element end forces in the plane of the element

3 Compression-Free Model in the Plane of the Membrane Element

The deformation of the triangular element can be defined as the changes, $\Delta l_1, \Delta l_2, \Delta l_3$ from the lengths of the three sides in the non-stress element, l_{10}, l_{20}, l_{30}. The deformations of the side lengths are given from the nodal position vectors, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ shown in Figure 1(a) with geometric exactitude, as follows.

$$\begin{aligned} \Delta l_1 &= l_1 - l_{10} = \sqrt{(\mathbf{u}_3 - \mathbf{u}_2)\bullet(\mathbf{u}_3 - \mathbf{u}_2)} - l_{10} \\ \Delta l_2 &= l_2 - l_{20} = \sqrt{(\mathbf{u}_1 - \mathbf{u}_3)\bullet(\mathbf{u}_1 - \mathbf{u}_3)} - l_{20} \\ \Delta l_3 &= l_3 - l_{30} = \sqrt{(\mathbf{u}_2 - \mathbf{u}_1)\bullet(\mathbf{u}_2 - \mathbf{u}_1)} - l_{30} \end{aligned} \tag{1}$$

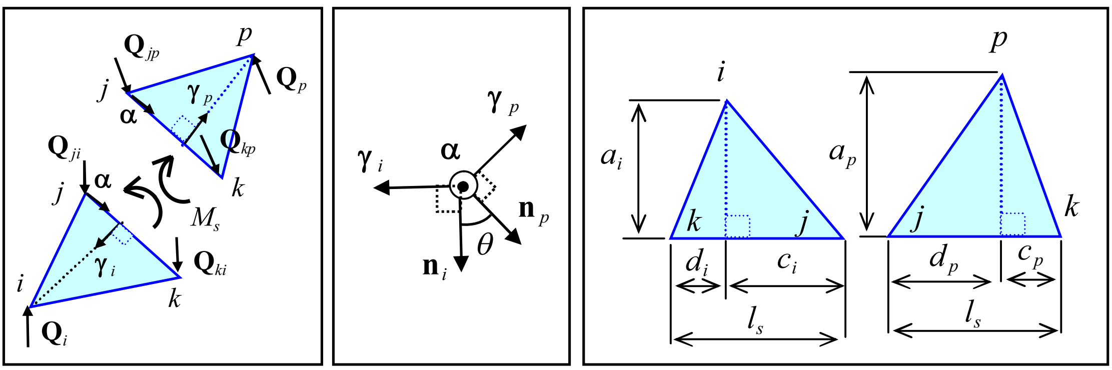
In the element of an orthotropic membrane, when the strain in the element is assumed to be uniform, the relation of the end force vector, $\boldsymbol{P_e} = \{P_1 \ \ P_2 \ \ P_3\}^{\mathrm{T}}$ and the elongations of the element sides, $\boldsymbol{\Delta l} = \{\Delta l_1 \ \ \Delta l_2 \ \ \Delta l_3\}^{\mathrm{T}}$ can be expressed by the following equation.

$$\mathbf{P}_e = \mathbf{a}^{\mathrm{T}}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{E}\boldsymbol{\omega}\mathbf{a}\Delta\mathbf{l} = \mathbf{k}_e\Delta\mathbf{l} \tag{2}$$

Where **E** is the matrix expressing the relation of the stress and the strain in the two directions of the fibers making a right angle. **ω** is the translation matrix based on the direction of the fiber in the triangular element and **a** is the translation matrix from the rectangular coordinates to the coordinates based on the direction of each side.

The method replaces the membrane element in uniform strain with the compression-free unit composed by 6 of axial members, as shown by Figure 1(b). The 6 of stiffness coefficients in $\mathbf{k}_e$ of Equation (2) provide the elongation stiffness per unit length, k_1, k_2, k_3 of the three members that form the element shape, the stiffness k_a of the three members connecting each apex and the sub-node, that position is also given by the coefficients in $\mathbf{k}_e$. In Figure 1(b), h_1, h_2 and h_3 are the distance between the sub-node and each side.

Hyperbolic function is applied to expressing the axial forces of the 6 members in the unit. The use keeps the compression-free characteristic from extremely slack state that the three apexes converge at a point to tense state with smoothly change of tensions.



(a) End shear forces balanced with the end moment (b) Direction vectors (c) Lengths in the two elements

Figure 2. Element end forces out of the plane of the element

4 End Shear Forces Based on the End Moment at the Element Side

Though bending rigidity of general membrane material is very small, considering the bending moment generated at a side of an element can prevent the element from shrinking occasionally into zero area because of the compression-free assumption. Moreover the consideration does not bring a multi-bifurcated problem.

The bending moment, M_s generating at the side can be converted into the two of the end moments working on each side. The sum of the rotational angles of the two ends by the end moments is defined as $\Delta\theta$. The rotational angle can be exactly given from the four positions of the apexes in the two elements. That is $\Delta\theta = \theta - \theta_0$, where θ_0 is the angle between the two planes of the elements in non-stress and θ is that of the two elements in a certain state of the structure. Using the direction vectors normal to the two planes, as shown in Figure 2(b), gives θ with geometric exactitude, as follows.

$$\cos\theta = \mathbf{n}_i \bullet \mathbf{n}_p, \qquad \sin\theta = \boldsymbol{\alpha} \bullet \mathbf{n}_i \times \mathbf{n}_p \tag{3}$$

When the two elements in Figure 2 are regarded as the two beams with the spans, a_{i0} and a_{p0}, that are the lengths of the perpendiculars in the two elements in non-stress, the widths of the two beams linearly change from zero to the side length, l_s along the perpendiculars. Therefore, the curvatures induced by the end moment, M_s are uniform along the perpendiculars on the assumption that the deformations of the two beams are very small, and then the relation of the end moment M_s and the changed angle $\Delta\theta$ becomes,

$$M_s = \frac{2EI_s}{a_{i0} + a_{p0}} \Delta\theta \tag{4}$$

In Equation (4), EI_s is the bending rigidity of the side jk, E is the Young's modulus in the direction normal to the side, and I_s is the moment of inertia of the cross section in the side.

5 Equilibrium Equations and the Tangent Stiffness Equations Including Geometric Stiffness

The nodal forces balanced with the end forces in the plane of the compression-free element shown in Figure 1(a) are defined as $\mathbf{U}_1^{\text{in}}$, $\mathbf{U}_2^{\text{in}}$, $\mathbf{U}_3^{\text{in}}$. In the universal coordinates, the end forces have the directions of the sides expressed by $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ and the equilibrium equations of those forces at the three nodes of the apexes become Equation (5).

$$\begin{Bmatrix} \mathbf{U}_1^{\text{in}} \\ \mathbf{U}_2^{\text{in}} \\ \mathbf{U}_3^{\text{in}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\alpha}_2 & -\boldsymbol{\alpha}_3 \\ -\boldsymbol{\alpha}_1 & \mathbf{0} & \boldsymbol{\alpha}_3 \\ \boldsymbol{\alpha}_1 & -\boldsymbol{\alpha}_2 & \mathbf{0} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \mathbf{C}_e^{\text{in}} \mathbf{P}_e \tag{5}$$

On the other hand, when the nodal forces balanced with the end shear forces shown in Figure 2(a) are defined as $\mathbf{U}_i^{\text{out}}$, $\mathbf{U}_j^{\text{out}}$, $\mathbf{U}_k^{\text{out}}$, $\mathbf{U}_p^{\text{out}}$, the equilibrium equation of those forces becomes the following.

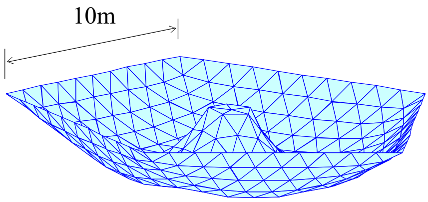
$$\begin{Bmatrix} \mathbf{U}_i^{\text{out}} \\ \mathbf{U}_j^{\text{out}} \\ \mathbf{U}_k^{\text{out}} \\ \mathbf{U}_p^{\text{out}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_i \\ \mathbf{Q}_{ji} + \mathbf{Q}_{jp} \\ \mathbf{Q}_{ki} + \mathbf{Q}_{kp} \\ \mathbf{Q}_p \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\mathbf{n}_i}{a_i} \\ \dfrac{d_i \mathbf{n}_i}{a_i l_s} + \dfrac{c_p \mathbf{n}_p}{a_p l_s} \\ \dfrac{c_i \mathbf{n}_i}{a_i l_s} + \dfrac{d_p \mathbf{n}_p}{a_p l_s} \\ -\dfrac{\mathbf{n}_p}{a_p} \end{Bmatrix} M_s = \mathbf{C}_s^{\text{out}} M_s \tag{6}$$

When the nodal forces working at the all of the nodes in the structure are defined as the force vector of $\mathbf{U}$ where $\mathbf{U}$ is the total sum of $\mathbf{U}_1^{\text{in}}$, $\mathbf{U}_2^{\text{in}}$, $\mathbf{U}_3^{\text{in}}$ in the all of the elements and $\mathbf{U}_i^{\text{out}}$, $\mathbf{U}_j^{\text{out}}$, $\mathbf{U}_k^{\text{out}}$, $\mathbf{U}_p^{\text{out}}$ in the all of the sides in the structure. Therefore, when the nodal forces balance with the end forces in the compression-free elements and the end shear forces at the all of the nodes, the unbalanced force vector of $\Delta\mathbf{U}$ in Equation (7) becomes zero vector.

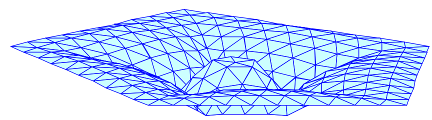
$$\mathbf{U} - \sum_{e=1}^{m_e} \mathbf{C}_e^{\text{in}} \mathbf{P}_e - \sum_{s=1}^{m_s} \mathbf{C}_s^{\text{out}} M_s = \Delta\mathbf{U} \tag{7}$$

where m_e is the total number of the elements in the structure and m_s is the total number of the sides.

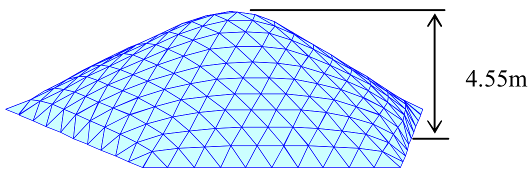
When the nodal forces are optionally given to the structure, the unbalanced forces in Equation (7) do not generally become zero. The method is to find the nodal positions fulfilling that the all of the unbalanced forces are less than the allowable value. Iterative methods such as the Newton-Raphson's method or the Dynamic Relaxation method can be applied to the computation.



(a) Initial shape of the membrane suspended from the circumference and the six nodes at the crown fixed.



(b) Shape under 13Pa in inner pressure; the six nodes at the crown are still fixed.



(c) Shape under 15Pa in inner pressure.

Figure 3. Inflating a membrane dome and the snapping phenomenon.

If the tangent stiffness equation shown by Equation (8) derived from differentiating the equilibrium equations of Equation (5) and Equation (6) is used in the iterative methods, the iteration quickly converges to the equilibrium solution.

$$\begin{aligned}\delta\mathbf{U} &= \sum_{e=1}^{m_e}\left(\delta\mathbf{C}_e^{\mathrm{in}}\mathbf{P}_e + \mathbf{C}_e^{\mathrm{in}}\delta\mathbf{P}_e\right) + \sum_{s=1}^{m_s}\left(\delta\mathbf{C}_s^{\mathrm{out}}M_s + \mathbf{C}_s^{\mathrm{out}}\delta M_s\right)\\ &= \left(\mathbf{K}_{\mathrm{G}}^{\mathrm{in}} + \mathbf{K}_{\mathrm{O}}^{\mathrm{in}}\right)\delta\mathbf{u} + \left(\mathbf{K}_{\mathrm{G}}^{\mathrm{out}} + \mathbf{K}_{\mathrm{O}}^{\mathrm{out}}\right)\delta\mathbf{u},\end{aligned} \tag{8}$$

6 Computational Example of Inflating a Membrane Structure

Figure 3 is a computational result of inflating a membrane dome with the pentagonal boundary fixed. The membrane used in the computation is an isotropic material, the elongation stiffness: E_t = 882kN/m, the Poisson ratio: v = 0.4 and the weight per unit area: w = 9.8N/m^2, where t is the thickness of the membrane. Those material coefficients roughly correspond to a membrane made from polyester textile coated with vinyl chloride. Incidentally, the snapping phenomena of membrane domes occur during the inflation without regard to isotropic material or orthotropic one. The dome is a three-dimensional structure formed by seaming the five sheets of the membrane of the isosceles triangle with the base of 10m and with the angle of 58° at the vertex in shape. In the computation, 500 elements of the triangle similar to the shape of the sheet of the membrane compose the dome. The meshing keeps enough accuracy in the computation. The circumference of the dome is fixed through the inflation, and the six nodes at the crown are fixed at the position of the same level as the circumference until the inner pressure increases to 13Pa. The six nodes are released over the inner pressure of 13Pa.

The initial shape of the membrane structure before the inflation is the suspended form, as shown in Figure 3(a), calculated on the condition of fixing both of the circumference and the crown. The inflation proceeds by increasing the inner pressure of the dome. When the inner pressure reaches to 13Pa, the membrane forms into Figure 3(b). At that moment, the curvatures at the seams become very large and the tension in the membrane become also so large. When the phenomenon occurs in a real structure of membrane, the structure almost collapses by cracks at the seams. Even if the part of the crown is not fixed, the phenomenon occurs under smaller pressure than 13Pa.

While the inner pressure increases from 13Pa, the part of the crown is released from the fixing. When the inner pressure reaches 15Pa, the shape becomes Figure 3(c). The process of the deformation from Figure 3(b) to Figure 3(c) is just a snapping phenomenon, and we should avoid the phenomenon in a real membrane structure.

7 Summary

The paper proposed a method that can stably analyze any large deformation of membrane structures. The method is to find the nodal positions fulfilling the equilibrium equations at the nodes expressed by the external nodal forces and the end forces derived from the deformation of the elements. The end forces are the two kinds of that in the plane of the compression-free element and the end shear forces normal to the element. By using the method, the paper explained snapping phenomena occurring in the inflation of a membrane dome. Since the phenomena bring destruction of the structure, the prediction of the occurrence and the countermeasure are important.

References

[1] K. Ijima, H. Obiya and N. Kawasaki, Large displacement analysis of compression-free membrane structures replaced by elastic cable units, *International Conference on Textile Composites and Inflatable Structures, Structural Membranes*, (2009).

[2] K. Ijima, H. Obiya and K. Kido, An orthotropic membrane model replaced with line-members and the large deformation analysis, *International Conference on Textile Composites and Inflatable Structures, Structural Membranes*, (2011)

[3] R. Phaal and C.R. Calladine, A simple class of finite elements for plate shell problems, II: An element for thin shells with only translational degrees of freedom, International Journal Numerical Method Engineering, **35**(5), 979-996, (1992).

[4] E. Oñate and M. Cervera, Derivation of thin plate bending elements with one degree of freedom per node, Engineering Computations, **10**, 553-561, (1993).

[5] F.G. Flores and E. Onate, A rotation-free shell triangle for the analysis of kinked and branching shells, International Journal Numerical Method Engineering, **69**(7)**,** 1521-1551, (2007).

[6] M. Tomioka, K. Ijima and H. Obiya, Large deformation analysis of flexible membrane structure and the experiment, *International Conference on Textile Composites and Inflatable Structures, Structural Membranes*, (2015).