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New method for solving the bending problem of rectangular plates with mixed boundary conditions

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Abstract. A new method is used to solve the rectangular plate bending problem with mixed boundary conditions. The method overcomes the complicated derivation of the classical solution by Fourth-order differential problem into integrating question. Under uniform loading rectangular plate bending problem with one side fixed the opposite side half simply supported half fixed the other two sides free rectangular plate, one side simply supported the opposite side half simply supported half fixed the other two sides free rectangular plate is systematically solved. According to the actual boundary conditions of the rectangular plate, the corresponding characteristic equation can easily be set up. It is presented deflection curve equation and the numerical calculation. By compared the results of the equation to the finite element program, we are able to demonstrate the correctness of the method. So the method not only has certain theoretical value, but also can be directly applied to engineering practice.

1 Introduction

With the development of the construction industry, the diversity of building is in high demand. It is more and more important to calculate plates under various constraint conditions. So in this paper, applying a new method to solve the bending problem of rectangular plates with mixed boundary conditions .The method provides a simple and practical calculation method for engineering calculation of rectangular plates with mixed boundary conditions, and it has certain theoretical and engineering significance. The method is no need to consider the hypothesis of complex displacement function, which can be very easy to write the total potential energy of rectangular plate on mixed boundary conditions. According to the actual boundary conditions of rectangular plate, corresponding characteristic equation can be easily established, thereby to overcome the cumbersome derivation process in classic solution. Because this method is simple and program characteristics, especially which is more efficient to solve the other methods are not easy to solve problems.

2 Theoretical basis for the new method

Static equilibrium equation of bending plates [1-2]

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D} \tag{1}$$

Fourth-order differential problem is converted to integrating question due to the complicated derivation of the classical solution.

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According to the generalized virtual work principle [3], there is

$$\iiint_V \sigma_{ij}\delta e_{ij}dV=\iiint_V F_i\delta u_i dV+\iint_{S_p}\overline{p_i}\delta u_i dS+\iint_{S_u} p_i\delta\overline{u}_i dS \tag{2}$$

Or according to the known boundary displacement variation potential energy principle [4]

$$\iiint_V \delta A(e)dV=\iiint_V F_i\delta u_i dV+\iint_{S_p}\overline{p_i}\delta u_i dS+\iint_{S_u} p_i\delta\overline{u}_i dS \tag{3}$$

Outside surface Yu Gong zero variation principle [5-7]

$$\iint_{S_u} p_i\delta\overline{u}_i \mathrm{d}S=-\iint_{S_u}\overline{u}_i\delta p_i \mathrm{d}S \tag{4}$$

Substituting (3), then

$$\iiint_V \delta A(e)dV=\iiint_V F_i\delta u_i dV+\iint_{S_p}\overline{p_i}\delta u_i dS-\iint_{S_u}\overline{u}_i\delta p_i dS \tag{5}$$

According to formula (5) can be obtained

$$\Pi_{mp}=\iiint_V \delta A(e)\mathrm{d}V-\iiint_V F_i\delta u_i \mathrm{d}V-\iint_{S_p}\overline{p_i}\delta u_i \mathrm{d}S+\iint_{S_u}\overline{u}_i\delta p_i \mathrm{d}S \tag{6}$$

Type (6) is called total potential energy of mixed variable potential energy principle [8], or simply mixed total potential energy.

Take extreme variation for u_i and p_i in the formula (6), then we have

$$\delta \prod_{mp} = \iiint_V \delta A(e) \mathrm{d}V - \iiint_V F_i \delta u_i \mathrm{d}V - \iint_{S_p} \overline{p_i} \delta u_i \mathrm{d}S + \iint_{S_u} \overline{u}_i \delta p_i \mathrm{d}S = 0 \qquad (7)$$

Pay attention to

$$\iiint_V \delta A(e) dV = \iiint_V \frac{\delta A(e)}{\partial e_{ij}} \delta e_{ij} dV = \iiint_V \sigma_{ij} \delta e_{ij} dV$$

$$= \iiint_V \sigma_{ij} \delta u_{i,j} dV = \iiint_V \left[\left(\sigma_{ij} \delta u_i \right)_{,j} - \sigma_{ij,j} \delta u_i \right] dV$$

$$= \iint_S \sigma_{ij} n_j \delta u_i dS - \iiint_V \sigma_{ij,j} \delta u_i dV$$

$$= \iint_{S_p} \sigma_{ij} n_j \delta u_i dS + \iint_{S_U} \sigma_{ij} n_j \delta u_i dS$$

$$- \iint_V \sigma_{ij,j} n_j \delta u_i dS \qquad (8)$$

Mixed total potential energy is required that the displacement is weakly allowed and the boundary force is coordination allowed [9]. According to this definition, the δu_i is not zero on S_u. Type formula (8) into (7), then

$$\delta \prod_{mp} = -\iiint_V \left(\sigma_{ij,j} + F_i \right) \delta u_i \mathrm{d}V + \iint_{S_P} \left(\sigma_{ij} n_j - \overline{p}_i \right)_i \delta u_i \mathrm{dS}$$
$$+ \iint_{S_u} \sigma_{ij} n_j \delta u_i \mathrm{d}S + \iint_{S_u} \overline{u}_i \delta p_i \mathrm{d}S = 0 \qquad (9)$$

According to the inner surface Yu Gong zero variation principle

$$\iint_{S_u} \sigma_{ij} n_j \delta u_i \mathrm{d}S = - \iint_{S_u} u_i \delta \left(\sigma_{ij} n_j \right) \mathrm{d}S \qquad (10)$$

Formula (9) becomes

$$\delta \prod_{mp} = -\iiint_V \left(\sigma_{ij,j} + F_i \right) \delta u_i \mathrm{d}V + \iint_{S_P} \left(\sigma_{ij} n_j - \overline{p}_i \right)_i \delta u_i \mathrm{dS}$$
$$- \iint_{S_u} \left(u_i - \overline{u}_i \right) \delta p_i \mathrm{d}S = 0 \qquad (11)$$

Taking extreme variation on the displacement and force can obtain the corresponding deflection curve equation.

3 One side fixed the opposite side half simply supported half fixed the other two sides free rectangular plate

Take only by the unit concentrated force simply supported rectangular plate as a basic system, as shown in Fig.1. To calculate the curved rectangular plate, taking one side fixed the opposite side half simply supported half fixed the other two sides free under uniform loads rectangular plate as the actual system, as shown in Fig.2.

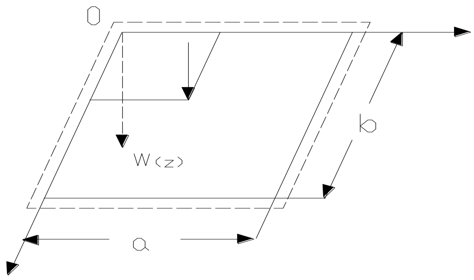

Figure 1. Virtual basic system for thin plate

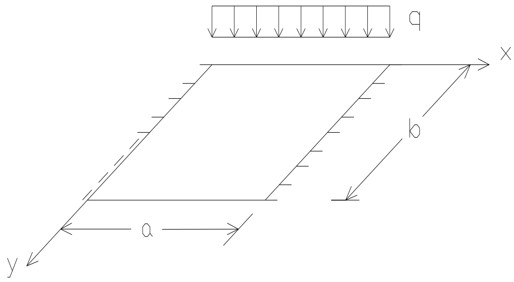

Figure 2. Actual system

3.1 Deflection surface equation

For the actual system Fig 2, fixed side bending constraint is released and this constraint is replaced by distributed moments, as shown in Fig 3.

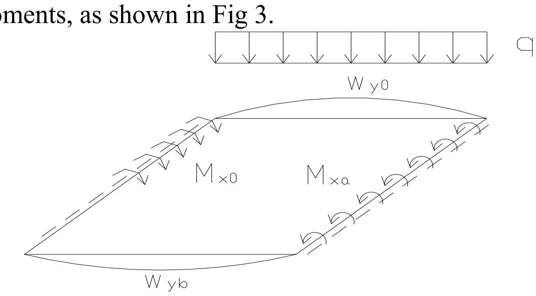

Figure 3. Actual system of fixed side bending constraints relesed

$$M_{x0} = \sum_{n=1,2}^{\infty} A_n \sin \beta_n y \qquad 0<y<\mathrm{b} \qquad (12)$$

$$M_{xa} = \sum_{n=1,2}^{\infty} B_n \sin \beta_n y \qquad 0<y<\mathrm{b}/2 \qquad (13)$$

y=0， y=b side deflection is:

$$W_{y0} = \sum_{n=1,2}^{\infty} C_m \sin \alpha_m x \qquad (14)$$

$$W_{yb} = \sum_{m=1,2}^{\infty} d_m \sin \alpha_m x \qquad (15)$$

The new method can get one side fixed the opposite side half simply supported half fixed the other two sides free rectangular plate total potential energy of minimum potential energy principle, such as type (16)

$$\prod_{mp}=\int_0^a\int_0^b\frac{1}{2}D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 dxdy-$$

$$\int_0^a\int_0^b\frac{2D(1-\nu)}{2}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]dxdy$$

$$-\int_0^a\int_0^b qwdxdy-\int_0^{\frac{b}{2}}\overline{M}_{x0}\left(\frac{\partial w}{\partial x}\right)_{x0}dy+$$

$$\int_0^b\overline{M}_{xa}\left(\frac{\partial w}{\partial y}\right)_{xa}dy+\int_0^a w_{yb}V_{yb}dx-\int_0^a w_{y0}V_{y0}dx \quad (16)$$

The formula (12) - (15) into equation (16), where taking $\prod_{mp}$ extreme variation to get deflection surface equation:

$$W(\xi,\eta)=\int_0^a\int_0^b qw_1\,dxdy+\int_0^{\frac{b}{2}}M_{x0}w_{1x0}dy-$$

$$\int_0^b M_{xa}w_{1xa}dy+\int_0^a V_{1y0}w_{y0}dy-\int_0^a V_{1yb}w_{yb}dx+R_{1ab}k_3$$

$$=w_1+w_2+w_3+w_4+w_5 \quad (17)$$

Among them:

$$w_1=\frac{4q}{Da}\sum_{m=1,3}^{\infty}\{1+\frac{1}{2ch\frac{1}{2}\alpha_m b}[\alpha_m(\eta-\frac{b}{2})sh\alpha_m(\eta-\frac{b}{2})$$

$$-(2+\frac{1}{2}\alpha_m bth\frac{1}{2}\alpha_m b)ch\alpha_m(\eta-\frac{b}{2})]\ \}\cdot\frac{1}{\alpha_m^5}\sin\alpha_m\xi$$

$$=\frac{4q}{Da}\sum_{n=1,3}^{\infty}\{1+\frac{1}{2ch\frac{1}{2}\beta_n a}[\beta_n(\zeta-\frac{a}{2})sh\beta_n(\zeta-\frac{a}{2})-$$

$$(2+\frac{1}{2}\beta_n ath\frac{1}{2}\beta_n a)ch\beta_n(\zeta-\frac{a}{2})]\ \}\cdot\frac{1}{\beta_n^5}\sin\beta_n\eta \quad (18)$$

$$w_2=\frac{1}{4D}\sum_{n=1,2}^{\infty}\left(-\frac{\beta_n a}{sh^2\beta_n a}sh\beta_n\xi+cth\beta_n a\beta_n\xi ch\beta_n\xi\right)$$

$$.\frac{A_n}{\beta_n^2}\sin\beta_n\eta-\frac{1}{4D}\sum_{n=1,2}^{\infty}\beta_n\xi sh\beta_n\xi\frac{A_n}{\beta_n^2}\sin\beta_n\eta \quad (19)$$

$$w_3=-\int_0^b M_{xa}w_{1xa}dy\frac{1}{2}\sum_{n=1,2}^{\infty}\left(\beta_n acth\beta_n a-\beta_n\xi cth\beta_n\xi\right)\cdot$$

$$\frac{B_n}{\beta_n^2 sh\beta_n a}sh\beta_n\xi\cdot\sin\beta_n\eta \quad (20)$$

$$w_4=\frac{1}{2}\sum_{n=1,2}^{\infty}2\frac{C_m}{sh\alpha_m b}sh\alpha_m(b-\eta)\sin\alpha_m\xi+$$

$$\frac{1}{2}\sum_{n=1,2}^{\infty}(1+\nu)\left[\alpha_m bcth\alpha_m b-\alpha_m(b-\eta)cth\alpha_m(b-\eta)\right]\cdot$$

$$\frac{C_m}{sh\alpha_m b}sh\alpha_m(b-\eta)\sin\alpha_m\xi \quad (21)$$

$$w_5=\frac{1}{2}\sum_{m=1,2}^{\infty}\left[2+(1-\nu)(\alpha_m bcth\alpha_m b-\alpha_m\eta cth\alpha_m\eta)\right]$$

$$\frac{d_m}{sh\alpha_m b}sh\alpha_m\eta\sin\alpha_m\xi \quad (22)$$

3.2 Boundary conditions

Deflection surface equation (18) must satisfy the following boundary conditions

$$\left(\frac{\partial w}{\partial\xi}\right)_{\xi=0}=0 \quad (23)$$

$$(w'_{1\xi}+w'_{2\xi}+w'_{3\xi}+w'_{4\xi}+w'_{5\xi})_{\xi=0}=0 \quad (24)$$

$$\left(\frac{\partial w}{\partial\xi}\right)_{\xi=a}=0 \quad (25)$$

$$(w'_{1\xi}+w'_{2\xi}+w'_{3\xi}+w'_{4\xi}+w'_{5\xi})_{\xi=a}=0 \quad (26)$$

$$-D\left[\frac{\partial^3 w}{\partial\eta^3}+(2-\nu)\frac{\partial^3 w}{\partial^2\xi\partial\eta}\right]_{\eta=0}=0 \quad (27)$$

$$-D[\omega'''_{1\eta\eta\eta}+\omega'''_{2\eta\eta\eta}+\omega'''_{3\eta\eta\eta}+\omega'''_{4\eta\eta\eta}+\omega'''_{5\eta\eta\eta}+(2-\nu)$$

$$.(\omega'''_{1\xi\xi\eta}+\omega'''_{2\xi\xi\eta}+\omega'''_{3\xi\xi\eta}+\omega'''_{4\xi\xi\eta}+\omega'''_{5\xi\xi\eta})]_{\eta=0}=0 \quad (28)$$

$$-D\left[\frac{\partial^3 w}{\partial\eta^3}+(2-\nu)\frac{\partial^3 w}{\partial^2\xi\partial\eta}\right]_{\eta=b}=0 \quad (29)$$

$$-D[\omega'''_{1\eta\eta\eta}+\omega'''_{2\eta\eta\eta}+\omega'''_{3\eta\eta\eta}+\omega'''_{4\eta\eta\eta}+\omega'''_{5\eta\eta\eta}+(2-\nu)$$

$$.(\omega'''_{1\xi\xi\eta}+\omega'''_{2\xi\xi\eta}+\omega'''_{3\xi\xi\eta}+\omega'''_{4\xi\xi\eta}+\omega'''_{5\xi\xi\eta})]_{\eta=b}=0 \quad (30)$$

Infinite equations are derived through taking the deflection surface equation into the corresponding boundary conditions of the formula (23) to formula (30) .It is obtained unknown number through assigning for equations.

3.3 Numerical calculation

Calculation parameters were taken as a=b=1m， q=100N/m^2, E=200GPa，h/a=0.005、0.1、0.2、0.3，v=0.3. Since the trigonometric convert to hyperbolic functions makes infinite series converges issue is resolved so m=n=1，2，3…50. By using MATLAB software programming for the A_n，C_m，B_n ，d_m and deflection expression. For example, h/a=0.3 compared with the calculation results of ANSYS, both tend to be consistent, the error is less than 5%.

Table 1. Under uniform loading for one side fixed the opposite side half simply supported half fixed the other two sides free rectangular plate deflection value

X	y=0.00	y=0.00	y=0.25	y=0.25	y=0.5	y=0.5	y=0.75	y=0.75	y=1	y=1
	Article	ANSYS	Article	ANSYS	Article	ANSYS	Article	ANSYS	Article	ANSYS
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.000340	0.000331	0.000396	0.000376	0.000672	0.000666	0.001426	0.001390	0.001501	0.001591
0.2	0.001115	0.001171	0.001234	0.001194	0.001563	0.001685	0.002538	0.002629	0.002796	0.002787
0.3	0.001956	0.001985	0.002122	0.002047	0.002487	0.002598	0.003469	0.003559	0.003886	0.003879
0.4	0.002582	0.002813	0.002773	0.002651	0.003079	0.003084	0.003978	0.004042	0.004197	0.004180
0.5	0.002895	0.002934	0.003082	0.002986	0.003289	0.003269	0.004186	0.004150	0.004365	0.004299
0.6	0.002581	0.002550	0.002699	0.002600	0.003036	0.002918	0.003896	0.003868	0.004162	0.004165
0.7	0.001897	0.001916	0.002054	0.001972	0.002465	0.002373	0.003419	0.003537	0.003836	0.003848
0.8	0.000997	0.001054	0.001196	0.001137	0.001585	0.001533	0.002491	0.002423	0.002787	0.002719
0.9	0.000328	0.000324	0.000428	0.000456	0.000679	0.000682	0.000139	0.000143	0.001569	0.001501
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

4 One side simply supported the opposite side half simply supported half fixed the other two sides free rectangular plate

To calculate the curved rectangular plate, taking one side simply supported the opposite side half simply supported half fixed the other two sides free under uniform loads rectangular plate as the actual system, shown in Fig.4.

4.1 Deflection surface equation

For the actual system Fig 4, fixed side bending constraint is released and this constraint is replaced by distributed moment, as shown in Fig 5.

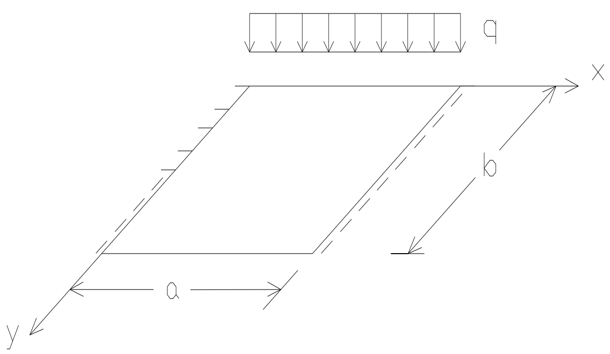

Figure 4. Actual system

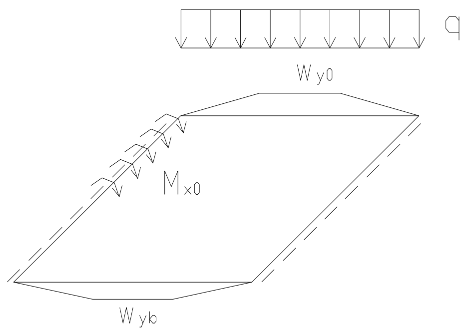

Figure 5. Actual system of fixed side bending constraints relesed

$$M_{x0} = \sum_{n=1,2}^{\infty} A_n \sin \beta_n y \tag{31}$$

y=0，y=b side deflection is:

$$w_{y0} = \sum_{,m=1,2}^{\infty} C_m \sin \alpha_m x \tag{32}$$

$$w_{yb} = \sum_{m=1,2}^{\infty} d_m \sin \alpha_m x \tag{33}$$

The new method can get one side simply supported the opposite side half simply supported half fixed the other two sides free rectangular plate total potential energy of minimum potential energy principle, such as type (34)

$$\prod_{mp}=\int_0^a\int_0^b\frac{1}{2}D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2dxdy-$$
$$\int_0^a\int_0^b\frac{2D(1-\nu)}{2}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]dxdy$$
$$-\int_0^a\int_0^b qwdxdy-\int_0^{\frac{b}{2}}\overline{M}_{x0}\left(\frac{\partial w}{\partial x}\right)_{x0}dy+$$
$$\int_0^a w_{yb}V_{yb}dx-\int_0^a w_{y0}V_{y0}dx \tag{34}$$

The formula (31) - (33) into equation (34), where taking $\prod_{mp}$ extreme variation to get deflection surface equation:

$$W(\xi,\eta)=\int_0^a\int_0^b qw_1dxdy+\int_0^{\frac{b}{2}}M_{x0}w_{1x0}dy+$$
$$\int_0^a V_{1y0}\omega_{yo}dx-\int_0^a V_{1yb}w_{yb}dx$$
$$=w_1+w_2+w_4+w_5 \tag{35}$$

w_1 is the same as type (18); w_2 is the same as type (19); w_4 is the same as type (21); w_5 is the same as the type (22).

4.2 Boundary conditions

Deflection surface equation (35) must satisfy the following boundary conditions

$$\left(\frac{\partial w}{\partial \xi}\right)_{\xi=0}=0 \tag{36}$$

$$-D\left[\frac{\partial^3 w}{\partial \eta^3}+(2-\nu)\frac{\partial^3 w}{\partial\eta\partial\xi^2}\right]_{\eta=0}=0 \tag{37}$$

$$-D\left[\frac{\partial^3 w}{\partial \eta^3}+(2-\nu)\frac{\partial^3 w}{\partial\eta\partial\xi^2}\right]_{\eta=b}=0 \tag{38}$$

Infinite equations are derived through taking the deflection surface equation into the corresponding boundary conditions of the formula (36) to formula (38) .It is obtained unknown number through assigning for equations.

4.3 Numerical calculation

Calculation parameters were taken a=b=1m，q=100N/m2, E=200GPa， h/a=0.005、0.1、0.2、0.3， v=0.3。m=n=1， 2， 3…50, by using MATLAB software programming for the An， Cm ,dm and deflection expression. For example, h/a=0.3 compared with the calculation results of ANSYS, both tend to be consistent, the error is less than 5%. The deflection curves of plate are showed in Fig.6.

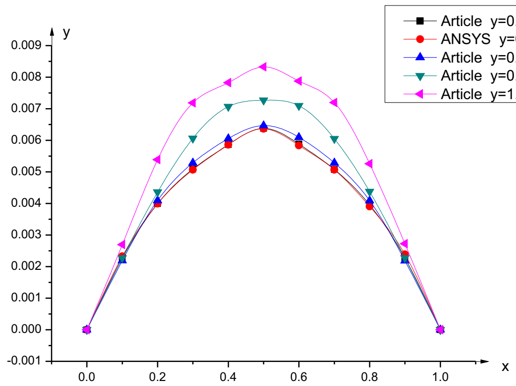

Figure 6. Virtual basic system for thin plate

5 Conclusion

In this paper, the new method is applied to solve the bending problem of rectangular plate with mixed boundary conditions under uniform loads. In the MATLAB platform, it is conducted numerical calculation for different issues. In addition, computational analysis of the rectangular plate is carried out using finite element software ANSYS. It is verified the accuracy of the numerical results by comparison, so that the new method is correct.

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