On the Adjacent Strong Equitable Edge Coloring of $P_n \lor P_n$, $P_n \lor C_n$, and $C_n \lor C_n$

Jun Liu\textsuperscript{1, a}, Chuan Cheng Zhao\textsuperscript{1}, Shu Xia Yao\textsuperscript{2}, Zhi Guo Ren\textsuperscript{1}, Qiu Ju Yue\textsuperscript{1}

1. Lanzhou City University, School of Information Science and Engineering, Lanzhou 730070, P.R. China
2. Lanzhou City University, Department of Mathematics, Lanzhou 730070, P.R. China

Abstract. A proper edge coloring of graph $G$ is called equitable adjacent strong edge coloring if colored sets from every two adjacent vertices incident edge are different, and the number of edges in any two color classes differ by at most one, which the required minimum number of colors is called the adjacent strong equitable edge chromatic number. In this paper, we discuss the adjacent strong equitable edge coloring of join-graphs about $P_n \lor P_n$, $P_n \lor C_n$, and $C_n \lor C_n$

1 Introduction

The coloring problem of graphs is widely applied in practice. In [1], some conditional coloring problems are introduced. Some network problem can be converted to practice. In [2-5], some conditional coloring problems as example.

DEFINITION 1 \textsuperscript{[2-5]} For a graph $G(V,E)$, if a proper coloring $f$ is satisfied with $C(u) \neq C(v)$ for $\forall u,v \in V(G)$, then $f$ is called $k$-strong edge coloring of $G$, which is abbreviated $k$-SEC, and

$$\chi'_s(G) = \min \{ k | k - \text{SEC of } G \}$$

is called the strong edge chromatic number of $G$. And for $\forall u,v \in E(G)$, $C(u) \neq C(v)$, then $f$ is called $k$-adjacent strong edge coloring of $G$, is abbreviated $k$-ASEC, and

$$\chi'_as(G) = \min \{ k | k - \text{ASEC of } G \}$$

called the adjacent strong edge chromatic number of $G$. Where

$$C(u) = \{ f(uv) | uv \in E(G) \}$$

DEFINITION 2 Let $f$ is a $k$-ASEC coloring of $G$, and satisfied with $|E_i - |E_i| \leq 1, i = 1, 2, \cdots, k$

$f$ is called the adjacent strong equitable edge coloring of $G$, and noted by $k$-ASEEC of $G$, and

$$\chi'_{as}(G) = \min \{ k | k - \text{ASEEC of } G \}$$

called the adjacent strong equitable edge chromatic number of $G$. Where

$$E_i = \{ e | f(e) = i \}$$

Conjecture \textsuperscript{[6]} For a connected graph with order $p \geq 3$ and $G \neq C_5$ (5-cycle),

Where $p = |V(G)|$, $\Delta(G)$ is maximal degree of $G$.

There are many references prove this conjecture is true, for example \textsuperscript{[7-8]}, for $\Delta(G) \leq 3$, this conjecture is true:

(1) If $G$ is a bipartite graph with no isolate edges, then

$$\chi'_{as}(G) \leq \Delta(G) + 2$$

(2) If $G$ is a $k$-chromatic graph with no isolate edges, then

$$\chi'_{as}(G) \leq \Delta(G) + O(\log k)$$

DEFINITION 3 \textsuperscript{[9]} For graph $G$ and graph $H$, $V(G) \cup V(H) = E(G) \cup E(H) = \emptyset$, and

$$(V(G) \cup H) = V(G) \cup V(H)$$

$$(E(G) \cup H) = E(G) \cup E(H) \cup \{ uv | u \in V(G), v \in V(H) \}$$

then $G \lor H$ is called join-graph of $G$ and $H$.

LEMMA 1 \textsuperscript{[6]} If $G$ is a connected graph with $|V(G)| \geq 3$, and $uv \in E(G), d(u) = d(v) = \Delta(G)$, then

$$\chi'_{as}(G) \geq \Delta(G) + 1$$

LEMMA 2 \textsuperscript{[6]} If $k \geq \chi'(G)$, then $k$-PEC of $G$ has been exist

$$|E_i - |E_i| \leq 1, i = 1, 2, \cdots, k$$

Where $e \in E_i, f(e) = i (i = 1, 2, \cdots, k), \chi'(G)$ is the

\textsuperscript{a}authore-mail: 527876625@qq.com

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LEMMA 3

For $n \geq 3$,
\[
|E(P_n \lor P_s)| = n^2 + 2n - 2
\]
\[
|E(P_n \lor C_s)| = n^2 + 2n - 1
\]
\[
|E(C_s \lor C_s)| = n^2 + 2n
\]

LEMMA 4

For $n \geq 3$, then
\[
x + y = n + 3
\]
\[
x + (n - 1)y = n^2 + 2n - 2
\]
Have a solution:
\[
x = 1
\]
\[
y = n - 2
\]

The equation
\[
x + y = n + 3
\]
\[
x + (n - 1)y = n^2 + 2n - 1
\]
Have a solution:
\[
x = 2
\]
\[
y = n + 1
\]

For $m > n \geq 1$, there are many adjacent strong chromatic number of $P_s \lor P_s, P_s \lor C_s$ and $C_s \lor C_s$. In this paper we have the adjacent strong equitable chromatic number of $P_s \lor P_s, P_s \lor C_s$ and $C_s \lor C_s$, the others terminologies refer to [9-10].

2 Adjacent Strong Edge Coloring of $P_n \lor P_n$

THEOREM 1

For $n \geq 2$, $\chi'_e(P_n \lor P_n) = n + 3$.

Proof There are four cases to be considered.

Case 1 When $n = 2$, then $P_2 \lor P_2 = K_4$ (complete graph with order 4), it’s true by [6].

When $3 \leq n \leq 5$, $\Delta(P_n \lor P_n) = n + 2$. By Lemma 1, we need to prove that exist a $n = 3$ –ASEEC.

Let $P_n$ and $P_n$ be $u_1u_2 \cdots u_n$ and $v_1v_2 \cdots v_n$.

Case 2 When $n = 3$, a mapping $f$ from $E(P_3 \lor P_3)$ to $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is defined as follows:
\[
f(u,v) = (f(u,v_1) = 1; f(u,v_2) = 2; f(u,v_3) = 3;
\]
\[
f(u_1v_1) = (f(u_1v_2) = 4; f(u_1v_3) = 5;
\]
\[
f(u_2v_1) = (f(u_2v_2) = 6.
\]

Obviously, $f$ is 8 –ASEEC of $P_3 \lor P_3$.

Case 3 When $n = 4$, a mapping $f$ from $E(P_4 \lor P_4)$ to $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is defined as follows:
\[
f(u,v) = i, i = 1, 2, 3, 4;
\]
\[
f(u_1v_1) = i + j + 1, i = 2, 3, j = 1, 2, 3, 4;
\]
\[
f(u_2v_1) = 5, f(u_2v_2) = 3;
\]
\[
f(u_3v_1) = 7; f(u_3v_2) = 1;
\]
\[
f(u_4v_1) = f(v_1v_2) = 7;
\]
\[
f(u_4v_2) = f(v_2v_3) = 1;
\]
\[
f(u_4v_3) = f(v_3v_4) = 2.
\]

For such $f$, we have:
\[
C(u_1) = \{5, 6\}; C(u_2) = \{3, 6\};
\]
\[
C(u_3) = \{2, 6\}; C(u_4) = \{3, 5\};
\]
\[
C(u_5) = \{2\}; C(u_6) = \{3\};
\]
\[
C(u_7) = \{6\}; C(v_1) = \{4\};
\]
\[
C(w) = \{1, 2, 3, 4, 5, 6, 7\}; C(w).
\]

Where
\[
w \in [u_1, u_2, u_3, u_4] \land [v_1, v_2, v_3, v_4].
\]

So $f$ is 7 –ASEEC of $P_4 \lor P_4$. The conclusion is true.

Case 4 When $n = 5$, a mapping $f$ from $E(P_5 \lor P_5)$ to $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is given as follows:
\[
f(u_1v_1) = 6; f(u_1v_2) = j, j = 2, 3, 4; f(u_1v_3) = 5;
\]
\[
f(u_2v_1) = i + 1, i = 2, 3, 4, 5;
\]
\[
f(u_3v_1) = 5; f(u_4v_2) = f(u_1v_2) = 7;
\]
\[
f(u_4v_3) = f(v_1v_2) = f(v_2v_3) = 8;
\]
\[
f(u_5v_1) = f(v_1v_2) = f(v_2v_3) = 1; f(u_5v_4) = 4;
\]
\[
f(u_5v_5) = f(v_1v_2) = 2; f(v_2v_3) = 3.
\]

For such $f$, we have:
\[
C(u_1) = \{1, 8\}; C(u_2) = \{i + 1\}, i = 2, 3, 4;
\]
\[
C(u_3) = \{3, 6\}; C(u_4) = \{1, 7\};
\]
\[
C(u_5) = \{4\}; C(v_1) = \{1, 2\};
\]
\[
C(v_2) = \{1, 2, 3, 4, 5, 6, 7, 8\} \lor C(w).
\]

Where
\[
w \in [u_1, u_2, u_3, u_4] \land [v_1, v_2, v_3, v_4].
\]

So $f$ is 8 –ASEEC of $P_5 \lor P_5$. The conclusion is true.

Case 5 When $n = 6$, a mapping $f$ from $E(P_6 \lor P_6)$ to $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is given as follows:
\[
f(u_1v_1) = 5; f(u_1v_2) = 7; f(u_1v_3) = 2;
\]
\[
f(u_2v_1) = 9; f(u_1v_4) = 4; f(u_1v_6) = 6;
\]
\[
f(u_3v_1) = 9; f(u_2v_2) = 8; f(u_2v_4) = 4;
\]
\[
f(u_4v_1) = 3; f(u_3v_3) = 6;
\]
\[
f(u_5v_1) = i + 1, i = 2, 3, 4, 5; j = 2, 3, 4, 5;
\]
\[
f(u_6v_1) = 8; f(u_5v_3) = 3;
\]
\[
f(u_6v_2) = 2; f(u_5v_5) = 1; f(u_5v_6) = 7;
\]
\[
f(u_6v_3) = 8; f(u_5v_7) = 9;
\]
\[
f(u_6v_4) = 3; f(u_5v_8) = 5;
\]
\[
f(u_6v_5) = f(u_5v_6) = f(v_1v_2) = f(v_2v_3) = 1;
\]
\[
f(u_6v_6) = f(u_5v_7) = f(v_3v_4) = 2;
\]
\[
f(u_6v_7) = 3; f(u_5v_8) = 4; f(v_1v_2) = 9.
\]

For the $f$, we have:
\[
C(u_1) = \{3, 8\}; C(u_2) = \{7\}; C(u_3) = \{3\};
\]
\[
C(u_4) = \{9\}; C(u_5) = \{1\}; C(u_6) = \{5, 9\};
\]
\[
C(v_1) = \{2, 7\}; C(v_2) = \{2\}; C(v_3) = \{8\};
\]
\[
C(v_4) = \{4\}; C(v_5) = \{5\}; C(v_6) = \{1, 4\};
\]

So, the $f$ is 9 –ASEEC of $P_6 \lor P_6$. The Theorem 2 is true when $n = 6$.

Case 6 When $n \geq 7$.

Subcase 6.1 When $n = 0(\text{mod} 2)$. Suppose $n = 2k, k \geq 2$.

From Lemma 5, $P_s \lor P_s$ exists a perfect matching $M_1;$
(P \lor P) \setminus \{u_i, u_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \);
(P \lor P) \setminus \{v_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \);

Subcase 6.2 When \( n = 1 (mod 2) \). Suppose \( n = 2k + 1, k \geq 3 \).

From Lemma 3, we can learn \( P \lor P \) exist a perfect matching \( M_1 \);
(P \lor P) \setminus \{u_i, u_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \);
(P \lor P) \setminus \{v_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \);

THEOREM 2 For \( n \geq 3 \),
\[
\mathcal{X}_m(P \lor C) = \begin{cases} 
3n+3, & \text{if } n = 0 (mod 2) \\
4n+4, & \text{if } n = 1 (mod 2) 
\end{cases}
\]

Proof If \( n = 0 (mod 2) \), suppose \( n = 2k, k \geq 2 \).
By lemma 1, we should only prove \( P \lor C \) exist \((n + 3)\) - \( ASSEE \).

By lemma 4, \( P \lor C \) exists two perfect matching \( M_1, M_2 \) and \( M_1 \setminus M_2 = \emptyset \).
(P \lor C) \setminus \{u_i, u_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \);
(P \lor C) \setminus \{v_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \);

\begin{align*}
\text{Subcase 6.2 When } n = 1 (mod 2). \text{ Suppose } n = 2k + 1, k \geq 3. \\
\text{From Lemma 3, we can learn } P \lor P \text{ exist a perfect matching } M_1; \\
(P \lor P) \setminus \{u_i, u_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \); \\
(P \lor P) \setminus \{v_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \); \\
(P \lor P) \setminus \{v_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \); \\
(P \lor P) \setminus \{u_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \); \\
(P \lor P) \setminus \{u_i, v_{i+1}\} \cup M_1, exist a perfect matching \( M_2 \); \\
\end{align*}

Let \( f \) be as follows
\[\forall e \in M_1, f(e) = i, i = 1, 2, \ldots, n + 3.\]

For \( f \) we have
\[\begin{align*}
C(u_i) &= (C(u_i)) = [i + 1], i = 1, 2, \ldots, k; \\
C(u_i) &= (C(u_i)) = [k + 2, n + 3]; \\
C(v_i) &= (C(v_i)) = [k + i + 1], i = 2, 3, \ldots, k; \\
C(v_i) &= (C(v_i)) = [k + 2, n + 3].
\end{align*}\]

And
\[|E| = \begin{cases} 
|n, i| = n - i + 1, i = 2, 3, \ldots, n. 
\end{cases}\]

So, \( f \) is a \((n + 3)\) - \( ASSEE \). \( P \lor C \)

For above reasons, the theorem 1 is true.

3 Adjacent Strong Edge Coloring of \( P \lor C \)
\[ \overline{C}(u_i) = \{n + 2i, 0\}; \overline{C}(u_i) = \{i - 1, n + i (mod(n + 4))\}, \]
\[ i = 2, 3, \ldots, n - 2; \]
\[ \overline{C}(u_{n-1}) = \{n - 3, 2n - 1 (mod(n + 4))\}; \overline{C}(u_i) = \{n - 1, n\}; \]
\[ \overline{C}(v_j) = \{n, n + 1, n + 3\}; \]
\[ \overline{C}(v_1) = \{n - 1, 2n - 1 (mod(n + 4)), 2n (mod(n + 4))\}; \]
\[ \overline{C}(v_1) = \{n + 1, n + 1, n + 4\}, \]
\[ n + i (mod(n + 4)), i = 2, 3, \ldots, n - 1. \]

So \( f \) is a \((n + 4) - \text{ASEEC} \) of \( P \) in \( C_n \).

Hence the conclusion is true.

4 Adjacent Strong Edge Coloring of \( C_n \vee C_n \)

**Theorem 3** For \( n \geq 3 \), then
\[
\chi''(C_n \vee C_n) = \begin{cases} 
  n + 3, & n \equiv 3 \pmod{2} \text{ or } n \equiv 0 \pmod{2} \\
  n + 4, & n \equiv 5 \pmod{2} \text{ or } n \equiv 1 \pmod{2} 
\end{cases}
\]

**Proof** Supposing the two cycles are \( u_4 u_5 \cdots u_k u_4 \) and \( v_1 v_2 \cdots v_y v_1 \) with separately.

When \( n = 3 \), \( C_n \vee C_n = K_3 \) (complete graph with order 6), can be seen in appendix.

**Case 1** \( n \equiv 0 \pmod{2} \), by lemma 2.4, 6, \( C_n \vee C_n \) exist three perfect matching \( M_1, M_2, M_3 \) and \( M_1 \cap M_2 = M_2 \cap M_3 = M_3 \cap M_1 = \emptyset \).

When \( n = 4 \), \( f \) be as follows:
\[
\begin{align*}
  f(u_{4i}) &= f(u_{4i+2}) = f(v_{4i}) = f(v_{4i+2}) = 7; \\
  f(u_{4i}) &= f(u_{4i+3}) = f(v_{4i}) = f(v_{4i+3}) = 12; \\
  f(u_{4i}) &= f(u_{4i+4}) = f(v_{4i}) = f(v_{4i+4}) = 13; \\
  f(u_{4i}) &= f(u_{4i+5}) = f(v_{4i}) = f(v_{4i+5}) = 14; \\
  f(u_{4i}) &= f(u_{4i+6}) = f(v_{4i}) = f(v_{4i+6}) = 23; \\
  f(u_{4i}) &= f(u_{4i+7}) = f(v_{4i}) = f(v_{4i+7}) = 34; \\
  f(u_{4i}) &= f(u_{4i+8}) = f(v_{4i}) = f(v_{4i+8}) = 44.
\end{align*}
\]

So \( f \) is a \( 7 - \text{ASEEC} \) of \( C_1 \vee C_1 \).

When \( n = 6 \), \( C_6 \vee C_n \) exist a perfect matching \( M_1 \).
\[
\begin{align*}
  (C_6 \vee C_6) \setminus \{u_{4i}, u_{4i+2}\} \cup M_1 \vee C_1; & \text{ exist a perfect matching } M_1; \\
  (C_6 \vee C_6) \setminus \{u_{4i}, u_{4i+3}\} \cup M_2; & \text{ exist a perfect matching } M_2; \\
  (C_6 \vee C_6) \setminus \{u_{4i}, u_{4i+4}\} \cup M_3; & \text{ exist a perfect matching } M_3; \\
  (C_6 \vee C_6) \setminus \{v_{4i}, v_{4i+2}\} \cup M_1; & \text{ exist a perfect matching } M_1; \\
  (C_6 \vee C_6) \setminus \{v_{4i}, v_{4i+3}\} \cup M_2; & \text{ exist a perfect matching } M_2; \\
  (C_6 \vee C_6) \setminus \{v_{4i}, v_{4i+4}\} \cup M_3; & \text{ exist a perfect matching } M_3. 
\end{align*}
\]

Let \( f \) be as follows:
\[
\forall v \in M_i, f(v) = i. 
\]

So \( f \) is a \( 9 - \text{ASEEC} \) of \( C_n \vee C_n \).

Similarly we can proof \( C_n \vee C_n \) exist \((n + 3) - \text{ASEEC}\) when \( n \equiv 0 \pmod{2} \) and \( n \geq 8 \).

By lemma 4, \( C_n \vee C_n \) exist three perfect matching \( M_i, M_2, M_3 \) and \( M_i \cap M_j = M_j \cap M_k = M_k \cap M_i = \emptyset \).

\[ \text{Suppose } n = 2k, k \geq 2. \]
\[ (C_n \vee C_n) \setminus \{u_{4i}, u_{4i+2}\} \cup M_{2i} \text{ exist a prefect matching } M_{2i}; \\
(C_n \vee C_n) \setminus \{u_{4i}, u_{4i+3}\} \cup M_{2i+1} \text{ exist a prefect matching } M_{2i+1}; \\
\]
\[ (C_n \vee C_n) \setminus \{u_{4i}, u_{4i+4}\} \cup M_{2i+2} \text{ exist a prefect matching } M_{2i+2}; \\
(C_n \vee C_n) \setminus \{v_{4i}, v_{4i+2}\} \cup M_{2i} \text{ exist a prefect matching } M_{2i}; \\
(C_n \vee C_n) \setminus \{v_{4i}, v_{4i+3}\} \cup M_{2i+1} \text{ exist a prefect matching } M_{2i+1}; \\
(C_n \vee C_n) \setminus \{v_{4i}, v_{4i+4}\} \cup M_{2i+2} \text{ exist a prefect matching } M_{2i+2}. 
\]

**Case 2** \( n \equiv 1 \pmod{2} \) and \( n \geq 5 \). We prove
\[ \chi''(C_n \vee C_n) \geq n + 4 \]

If \( \chi''(C_n \vee C_n) = n + 3 \), there are at least 3 colors represented at every vertex. For \( E(C_n \vee C_n) = n^2 + 2n \), and each vertex lack just one color and the vertices lack the same color in a same cycle and not adjacent, and mis an odd, so it is not possible that odd number vertices lack same color. For the edge number of \( C_n \vee C_n \), the number of vertices lack same color just an even. So there must have two vertices in two different cycle lack same color. It is contradictory. Thus \( \chi''(C_n \vee C_n) = n + 4 \) when \( n = 1 \pmod{2} \).

Let \( f \) be as follows:
\[
\begin{align*}
  (C_n \vee C_n) \setminus \{u_{4i}, u_{4i+2}, u_{4i+4}, \ldots, u_{4i + n - 4}\} & \text{ are coloring colors with } n + 3, n + 4 \text{ rotate, the edges of } \\
  v_{4i}, v_{4i+2}, \ldots, v_{4i + n - 4} & \text{ are coloring colors with } n + 4, n + 3 \text{ rotate; } \\
  f(u_{4i}) &= f(v_{4i}) = n + i; f(u_{4i+2}) = j, f = 1, 2, \ldots, n; \\
  f(u_{4i+4}) &= j + f(\text{when } i + j > n + 2), \\
  \text{then } n + 2, i = 2, 3, \ldots, n - 1; j = 1, 2, \ldots, n; \\
  & f(u_{4i}) = f(u_{4i+2}) = f(\text{when } i + j > n + 2), \\
  \text{take } n + 2, i = 2, 3, \ldots, n - 1; j = 1, 2, \ldots, n. 
\end{align*}
\]

For the \( f \), we have
\[
\begin{align*}
  \overline{C}(u_i) &= \{n + 2, n + 4\}; \overline{C}(u_i) = \{i - 1, i\}, \\
  i &= 2, 3, \ldots, n - 1; \overline{C}(u_i) = \{n, n + 3\}; \overline{C}(u_i) = \{1, n + 2\}; \\
  \overline{C}(u_i) &= \{2, n + 3\}; \overline{C}(u_i) = \{i + 1, n + i\} + \{\text{when } i + j > n + 2, \\
  \text{take } n + 2, \overline{C}(u_i) = \{n, n + 4\}; \overline{C}(u_i) = \{1, n + 2\}. 
\end{align*}
\]

So, \( f \) is a \( n + 4 - \text{ASEEC} \) of \( C_n \vee C_n \), the theorem 3 is true.

From all of above, the theorem 3 is true.

**References**