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Some Equal Degree Graph Edge Chromatic Number

Jun Liu[1,a], Zhi Guo Ren[1], Qiu Ju Yue[1], Zhong Yi Feng[1] and Cai Hui Lan[1]

1. Lanzhou City University, School of Information Science and engineering, Lanzhou 730070, P.R.China

Abstract. Let $G(V, E)$ be a simple graph and k is a positive integer, if it exists a mapping of f, and satisfied with $f(e_1) \neq 6=f(e_2)$ for two incident edges $e_1,e_2 \in E(G)$, $f(e_1) \neq 6=f(e_2)$, then f is called the k-proper-edge coloring of G(k-*PEC* for short).The minimal number of colors required for a proper edge coloring of G is denoted by $X'(G)$ and is called the proper edge chromatic number. It exists a k-proper edge coloring of simple graph of G, for any two adjacent vertices u and v in G, the set of colors assigned to the edges incident to u differs from the set of colors in cident to v, then f is called k-adjacent-vertex-distinguishing proper edge coloring, is avvreviated k-*AVDEC*, also called a adjacent strong edge coloring. The minimal number of colors required for a adjacent-vertex-distinguishing edge coloring of G is denoted by $X'_{ad}(G)$ and is called adjacent-vertex-distinguishing edge chromatic number. The new class graphs of equal degree graph are be introduced, and this class graphs adjacent-vertex-distinguishing edge chromatic numbers of path ,cycle, fan, complete graph, wheel, star are presented in this paper.

1 Introduction

The coloring problem of graphs is a extremely difficult problem, widely applied in practice. In [1], some conditional coloring problems are introduced. Some network problems can be converted to the edge coloring[2−9] and adjacent strong edge coloring.

Definition 1[10] For a graph$G(V, E)$, if a proper coloring f is satisfied with $C(u) \neq C(v)$ for $u,v \in V(G)(u \neq v)$ and $uv \in E(G)$ then f is called k-adjacent-vertex- distinguishing edge coloring of G, is abbreviated k-*AVDEC*, and

$$X'_{ad}(G) = \min\{k \mid k - AVDECofG\}$$

is called the adjacent-vertex-distinguishing edge chromatic number of G[6], where $C(u) = \{f(uv) \mid uv \in E(G)$

Conjecture[2] Let G be a connected graph with $|G| \geq 3$, and $G \neq C_5(5-cycle)$, then

$$X'_{ad}(G) \leq \Delta(G)+2$$

where $\Delta(G)$ is the maximum degree of graph G.

Definition 2[5] Let$G(V, E)$ be a simple graph, $M(G)$ is called the equal degree graph of G, where

$$V(M(G)) = V(G) \cup V' \cup \{W\};$$

$$E(M(G)) = E(G) \cup \{uv' \mid u \in V(G),$$
$$v' \in V', d(u) = d(v)\} \cup \{wv' \mid v' \in V'\}$$

Where as

$$V' = \{v' \mid v \in V(G)\}, \{w\} \cap (V(G) \quad \vee V') = \varphi.$$

2 Main result

Lemma 1 For $n \geq 2$, then

$$\Delta(M(P_n)) = \begin{cases} 3, n = 2,3; \\ n, n \geq 4. \end{cases}$$

Lemma 2 For $n \geq 3$, then

$$\Delta(M(P_n)) = n+2$$

Lemma 3 For $n \geq 3$, then

$$\Delta(M(F_n)) = \begin{cases} 5, n = 2,3; \\ n+1, n \geq 4. \end{cases}$$

Lemma 4 For $n \geq 2$, then

$$\Delta(M(K_n)) = \begin{cases} 3, n = 2 \\ 5, \ N = 3; \\ 2n-1 \geq 4. \end{cases}$$

Lemma 5 For $n \geq 3$, then

$$\Delta(M(W_n)) = \begin{cases} 7, n = 3; \\ n+3, n \geq 4. \end{cases}$$

[a]authore-mail: 527876625@.qq.com
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Lemma 6 For $n \ge 3$, then

$$\Delta(M(S_n)) = \begin{cases} 3, n = 1,2; \\ n+1, n \ge 3. \end{cases}$$

Lemma7 [2] Let G be a connected simple graph with$|V(G)| \ge 3$, if $uv \in E(G)$ and $d(u) = d(v) = \Delta(G)$, then

$$X'_{ad}(G) \ge \Delta(G) + 1$$

Theorem 1 For $n \ge 2$, then

$$X'_{ad}(M(P_n)) = \begin{cases} 5, n = 3; \\ n+1, n \ge 4. \end{cases}$$

Proof Let $V(P_n) = \{u_1, u_2, \cdots, u_n\}, V' = \{v_1 v_2, \cdots, v_n\}$, there are two cases to be discussed as follow:

Case1 When n= 2, $\Delta(M(P_2))$ = 3 from lemma 1. Howeveru_1, u_2, v_1, v_2 are vertices which are maximum degree of $M(P_2)$ and all adjacent, X'_{ad} $(M(P_2)) \ge \Delta(M(P_n))$ = 4 from lemma 7. If $M(P_2)$ exists *4-AVDEC*, because four vertices of maximum degree are adjacent each other, and $\binom{4}{3}$ = 4, the set are {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}.Let every set correspond a vertex, we can obtain the vertices of coloring same color are odd, but the vertices of coloring same color are even by graph theory, so $M(P_2)$ don't exists 4-AVDEC. In order to prove the conclusion be true, we only need construct a map f from $E(M(P_2))$ to {1,2,3,4,5}:

$f(u_1v_2) = f(u_2v_1) = 1;$

$f(u_1v_1) = 2;$

$f(u_2v_2) = 3;$

$f(u_1u_2) = 4;$

$f(wv_1) = 3;$

$f(wv_2) = 5$

Then we have

$\overline{C}(u_1) = \{3,5\};$

$\overline{C}(u_2) = \{2,5\};$

$\overline{C}(v_1) = \{4,5\};$

$\overline{C}(v_2) = \{2,4\};$

Obviously, the f is *5-AVDEC* of $M(P_2)$, the conclusion is true.

When n= 3, $\Delta(M(P_3))$ = 3 from lemma 1. But $M(P_2) \subset M(P_3)$,$M(P_3) \ge 5$ from n=2. It is obviously to prove exists *5-AVDEC* of $M(P_3)$, we only need to construct a map f from $E(M(P_3))$ to {1,2,3,4,5}:

$f(u_1u_2) = f(u_3v_3) = 1;$

$f(u_1v_1) = f(u_2u_3) = 2;$

$f(u_1v_3) = f(wv_2) = 3;$

$f(u_1uv_2) = f(wv_1) = 4;$

$f(u_3v_1) = f(wv_3) = 5;$

Then we have

$\overline{C}(u_1) = \{4,5\};$

$\overline{C}(u_2) = \{3,5\};$

$\overline{C}(u_3) = \{3,4\};$

$\overline{C}(v_1) = \{1,3\};$

$\overline{C}(v_2) = \{1,2,5\};$

$\overline{C}(v_3) = \{2,4\};$

Obviously, the f is *5-AVDEC* of $M(P_3)$, the conclusion is true.

Case2 When $n \ge 4$, $\Delta(M(P_n))$ = n from lemma 1. However the vertices of u_2, u_3, $\cdots$,u_{n-1}, w are vertices which are maximum degree, and exist some are adjacent, X'_{ad} $(M(P_n))$ $\ge$ $n+1$ from lemma 7. It is obviously to prove exists ($n+1$)-*AVDEC* of $M(P_n)$, we only need to construct a map f from $E(M(P_n))$ to {1,2,· · ·, n,0}:

$f(u_iu_{i+1}) = i-1, i = 1,2,\cdots,n-1;$

$f(u_1v_1) = 1; f(u_1v_n) = 2;$

$f(u_iv_j) = i + j - 2(\bmod n),$

$i = 2,3,n-1; j = 2,3,n-1;$

$f(u_nv_1) = n; f(u_nv_n) = 0;$

$f(wv_i) = i-1, i = 1,2, \cdots, n.$

So we have

$\overline{C}(v_i) = \{0,1,2\};$

$\overline{C}(u_2) = \{n\};$

$\overline{C}(u_i) = \{i-3\}, i = 3,4,\cdots,n-1;$

$\overline{C}(u_n) = \{0, n-2, n-1\};$

$\overline{C}(v_1) = \{0,1,n\};$

$\overline{C}(v_i) = \{i-1, i, \cdots, n+i-3\}(\bmod n+1),$

$i = 2,3,\cdots,n-1;$

$\overline{C}(v_n) = \{0,2,n-1\}.$

Obviously, the f is ($n+1$)-*AVDEC* of $M(P_n)$, the conclusion is true.

From all of above two cases, the conclusion is true.

Theorem 2 For $n \ge 3$, then

$$X'(M(C_n)) = n+3.$$

Proof Let $V(C_n) = \{u_1, u_2, \cdots, u_n\}, V' = \{v_1, v_2, \cdots, v_n\}, u_1, u_2, \cdots, u_n$ are vertices of maximum degree andexists some adjacent, $X'_{ad}(M(C_n)) \ge n+3$ from lemma 2 and 7. It is obviously to prove exists ($n+3$)-*AVDEC* of$\ge M(C_n)$, we only need to construct a map f from $E(M(C_n))$ to {1,2,· · ·, n+2,0}:

$f(u_iu_{i+1}) = i-1, i = 1,2,\cdots,n-1;$

$f(u_1u_n) = 0;$

$f(u_iv_j) = i + j(\bmod n+3),$

$i = 1,2,\cdots,n-1; j = 1,2,\cdots n;$

$f(u_nv_j) = n + j, j = 1,2;$

$f(u_nv_j) = j-2, j = 3,4,\cdots,n;$

$f(wv_j) = j-1.$

So we have

$\overline{C}(u_1) = \{n+2\};$

$\overline{C}(u_i) = \{i-1\}, i = 2,3,\cdots,n-1;$

$\overline{C}(u_n) = \{n\}.$

Obviously, the f is (n+3)-*AVDEC* of $M(C_n)$, the conclusion is true.

Theorem 3 For $n \geq 3$, then

$$X'(M(F_n)) = \begin{cases} 6, n = 2,3; \\ n+2, n \geq 4. \end{cases}$$

Proof Let $V(F_n) = \{u_0, u_1, u_2, \cdots, u_n\}$, $V' = \{v_0, v_1, v_2, \cdots, u_n\}$, there are three cases to be discussed as follow:

Case1 When n=2,$M(F_2)$ =$M(C_3)$, so we can obtain $X'(M(F_3)) = X'(M(C_3)) = 6$, then the conclusion is true.

Case2 When n= 3, $\Delta(M(F_3))$ = 5 from lemma 3, and $M(F_3)$ exists two vertices of maximum degree are adjacent, so X'_{ad} $(M(F_3))$ $\geq$ 6 by lemma 7. It is obviously to prove exists 6-*AVDEC* of $M(F_3)$, we only need to construct a map f from $E(M(F_3))$ to $\{1,2,\cdots,5,0\}$:

$f(u_0u_i) = i-1, i = 1,2,3;$

$f(u_iu_{i+1}) = i+1, i = 1,2;$

$f(u_0v_0) = 3;$

$f(u_0v_2) = 4;$

$f(u_1v_1) = 3;$

$f(u_3v_1) = 4;$

$f(u_3v_3) = 5;$

$f(wv_i) = i-1, i = 0,1,2,3.$

The we have

$\overline{C}(u_0) = \{5\};$

$\overline{C}(u_2) = \{0\}.$

Obviously, the f is 6-*AVDEC* of $M(F_3)$, the conclusion is true.

Case3 When $n \geq 4$, $\Delta(M(F_n))$ = n+ 1 from lemma 3,$M(F_n)$ exists some vertices of maximum degree are adjacent, X' $(M(F_n))$ $\geq \Delta(M(F_n))$+1=n+2 by lemma 3,7. It is obviously to prove exists (n+2)-*AVDEC* of $M(F_n)$, we only need to construct a map f from $E(M(F_n))$ to $\{1,2,\cdots, n+1,0\}$:

$f(u_0u_i) = i-1, i = 1,2,\cdots,n;$

$f(u_iu_{i+1}) = i+1, i = 1,2\cdots,n-1;$

$f(u_0v_0) = n;$

$f(u_1v_1) = 3;$

$f(u_1v_n) = 4;$

$f(u_iv_j) = i + j (\bmod n+2),$

$i = 2,3,\cdots,n-1; j = 2,3,\cdots,n-1;$

$f(u_nv_1) = 1;$

$f(u_nv_n) = 2;$

$f(wv_i) = i+1, i = 0,1,\cdots,n.$

So we have

$\overline{C}(u_0) = \{n+2\};$

$\overline{C}(u_i) = \{i-2\}, i = 2,3,\cdots,n-1;$

$\overline{C}(w) = \{0\}.$

Obviously, the f is (n+2)-*AVDEC* of $M(F_n)$, the conclusion is true.

From all of above, the conclusion is true.

Theorem 4 For $n \geq 2$, then

$$X'(M(K_n)) = \begin{cases} 5, n = 2; \\ 6, n = 3; \\ \Delta(M(K_n)) + 1, n \geq 4. \end{cases}$$

Proof Let $V(F_n) = \{u_1, u_2, u_2, \cdots, u_n\}, V' = \{v_1, v_2, \cdots, u_n\}$, there are three cases to be discussed as follow:

Case 1 When n= 2,$M(K_2)$ =$M(P_2)$, so we can obtain X' $(M(K_2))$ = X' $(M(P_2))$ = 5, the conclusion is true.

Case 2 When n= 3,$M(K_3)$ =$M(C_3)$, so we can obtain X' $(M(K_3))$ = X' $(M(C_3))$ = 6, the conclusion is true.

Case 3 When $n \geq 4$, $\Delta(M(K_n))$ = 2n−1 from lemma 4,$M(K_n)$ exists some vertices of maximum degree are adjacent, X' $(M(K_n)) \geq \Delta(M(K_n))$+1=2$n$ by lemma 4,7. It is obviously to prove exists 2n-*AVDEC* of $M(K_n)$,we only need to construct a map f from $E(M(K_n))$ to $\{1,2,\cdots,2n-1,0\}$:

$f(u_iu_j) = i + j,$

$i = 1,2,\cdots,n-1; j = i+1, i+2\cdots,n-1;$

$f(u_iv_j) = i + j + n - 3(\bmod 2n),$

$i = 1,2,\cdots,n; j = 1,2,\cdots,n-1;$

$f(v_nu_1) = 2n-1;$

$f(v_nu_i) = 2(i-1)-1, i = 2,3,\cdots,n;$

$f(wv_i) = i, i = 1,2,\cdots,n-1;$

$f(wv_n) = n, n \equiv 0(\bmod 2);$

$f(wv_n) = n+1, n \equiv 1(\bmod 2).$

The we have

$$\overline{C}(u_i) = \{2n+i-3\}(\bmod 2n), i = 1,2,\cdots,n.$$

Obviously, the f is 2n-*AVDEC* of $M(K_n)$, the conclusion is true.

From all of above, the conclusion is true.

Theorem 5 For $n \geq 3$, then

$$X'(M(W_n)) = \begin{cases} 8, n = 3; \\ n+4, n \geq 4. \end{cases}$$

Proof Let $V(W_n) = \{u_0, u_1, u_2, \cdots, u_n\}, V' = \{v_0, v_1, v_2 \cdots, u_n\}$, there are two cases to be discussed as follow:

Case1 When n= 3,$M(W_3)$ =$M(K_4)$, so we can obtain X' $(M(W_3))$ = X' $(M(K_4))$ = 8, the conclusion is true.

Case2 When $n \geq 4$, $\Delta(M(W_n))$ = n+ 3 from the lemma 5,$M(W_n)$ exists some vertices of maximum degree are adjacent, X' $(M(W_n))$ $\geq$ $\Delta(M(W_n))$+1=n+4 by lemma 5,7. It is obviously to prove exists (n+4)-*AVDEC* of $M(W_n)$, we only need to construct a map f from $E(M(W_n))$ to $\{1,2,\cdots, n+3,0\}$:

$f(u_0u_i) = i-1, i = 1,2,\cdots,n;$

$f(u_0v_0) = n;$

$f(u_iv_j) = i + j - 1(\bmod n+4),$

$i = 1,2,\cdots,n-1; j = 1,2,\cdots n;$

$f(u_nv_1) = n;$

$f(u_nv_j) = n + j(\bmod n+4), j = 2,3,\cdots,n-1;$

$f(u_nv_n) = n-3;$

$f(u_iu_{i+1}) = n+i+1(\bmod n+4),$
$i = 1,2,\cdots,n-1;$
$f(u_1u_n) = n+1;$
$f(wv_i) = n+2+j(\bmod n+4), i = 0,1,\cdots n,.$

Then we have

$\overline{C}(u_i) = \{n+i+2\}(\bmod n+4), i = 1,2,\cdots,n.$

Obviously, the f is (n+3)-*AVDEC* of $M(W_n)$, the conclusion is true.

From all of above, the conclusion is true.

Theorem 6 For $n \geq 1$, then

$$X'(M(S_n)) = \begin{cases} 5, n = 1,2; \\ n+3, n \geq 3. \end{cases}$$

Proof Let $V(S_n) = \{u_0, u_1, u_2, \cdots, u_n\}$, $V' = \{v_0, v_1, v_2 \cdots, u_n\}$,there are three cases to be discussed as follow:

Case 1 When n=1,$M(S_1)$=M(P_2), so we can obtain X' ($M(S_1)$)= X' ($M(P_2)$)=5, the conclusion is true.

Case 2 When n=2,$M(S_2)$=$M(P_3)$, so we can obtain X' ($M(S_2)$)= X' ($M(P_3)$)=6, the conclusion is true.

Case 3 When $n \geq 3$, $\Delta(M(S_n))$=n+1 by lemma 6,$M(S_n)$ exists some vertices of maximum degree are adjacent, X' ($M(S_n)$) $\geq \Delta(M(S_n))$+1=n+2 by lemma 6,7. If $M(S_n)$ exists n+2-*AVDEC*, then

$\overline{C}(u_0) = \overline{C}(u_i) = \overline{C}(v_i) = 1, i = 1,2,\cdots n.$

$$\begin{cases} \overline{C}(u_0) \in \overline{C}(u_1) \cup \overline{C}(u_2) \cup \cdots \cup \overline{C}(u_n); \\ \overline{C}(u_1) \in \overline{C}(v_1) \cup \overline{C}(u_2) \cup \cdots \cup \overline{C}(v_n) \\ \overline{C}(u_2) \in \overline{C}(v_1) \cup \overline{C}(v_2) \cup \cdots \cup \overline{C}(v_n) \\ \cdots \\ \overline{C}(u_n) \in \overline{C}(v_1) \cup \overline{C}(v_2) \cup \cdots \cup \overline{C}(v_n) \end{cases}$$

from (1) and (2), we can obtain

$|\overline{C}(u_0) + \overline{C}(u_1) + \overline{C}(u_2) + \ldots + \overline{C}(u_n) + \overline{C}(v_1) + \overline{C}(v_2) + \ldots + \overline{C}(v_n)| > n+2.$

It is obvious that $M(S_n)$ don't exists n+2-*AVDEC*. In order to prove the conclusion be true, we only need to construct a map f from $E(M(S_n))$ to {1,2,· · ·, n+2,0}:

$f(u_0u_i) = i-1, i = 1,2,\cdots,n;$
$f(u_0v_0) = n;$
$f(u_iv_j) = i+j(\bmod n+3),$
$i = 1,2,\cdots,n; j = 1,2,\cdots n;$
$f(wv_i) = i-1, i = 0,1,\cdots,n$

Then we have

$\overline{C}(u_0) = \{n+1, n+2\};$
$\overline{C}(u_i) = \{1, n+2\};$
$\overline{C}(u_i) = \{i-2, i\}, i = 2,3,\cdots,n;$
$\overline{C}(v_1) = \{0, n+2\};$
$\overline{C}(v_i) = \{i-2, i-1\} i = 2,3,\cdots n;$

Obviously, the f is (n+3)-*AVDEC* of $M(S_n)$, the conclusion is true.

From all of above, the conclusion is true.

From the conclusions of Theorem 1,2,3,4,5,6, the Conjecture is true for equal degree graph.

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