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On path hypercompositions in graphs and automata

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Abstract. The paths in graphs define hypercompositions in the set of their vertices and therefore it is feasible to associate hypercompositional structures to each graph. Similarly, the strings of letters from their alphabet, define hypercompositions in the automata, which in turn define the associated hypergroups to the automata. The study of the associated hypercompositional structures gives results in both, graphs and automata theory.

1 Introduction

An *operation* or *composition* in a non-void set H is a function from $H\times H$ to H while a *hyperoperation* or *hypercomposition* is a function from $H\times H$ to the power set $\mathscr{P}(H)$ of H. An algebraic structure that satisfies the axioms:

i. $a(bc) = (ab)c$ for every $a,b,c \in H$ (associativity)
ii. $aH = Ha = H$ for every $a \in H$ (reproductivity)

is called *group* if «·» is a composition, and *hypergroup* [6] if «·» is a hypercomposition [13]. A set H endowed with a hypercomposition "·" is called *hypergroupoid* if $xy\neq\varnothing$ for all x, y in H, otherwise it is called *partial hypergroupoid*. If A and B are non-empty subsets of H, then AB signifies the union $\bigcup_{(a,b)\in A\times B} ab$. If $A=\varnothing$ or $B=\varnothing$, then $AB=\varnothing$. Ab and aB will have the same meaning as $A\{b\}$ and $\{a\}B$. A hypergroupoid is called *semi-hypergroup* if only (i) is valid, while it is called *quasi-hypergroup* if only (ii) holds. A hypercomposition is called *closed* (or *containing*) if the two participating elements are included in the result. A hypercomposition is called *right closed* if $a \in ba$ for all $a,b \in H$ and *left closed* if $a \in ab$ for all $a,b \in H$. A hypercomposition is called *right open* if $a \notin ba$ for all $a,b \in H$ with $b \neq a$. The definition of *left open* hypercomposition is similar. Obviously, a hypercomposition is *open* if it is both right and left open. In [13] the following propositions has been proved:

Proposition 1. *The hypercomposition in a hypergroup H is right closed if and only if $a/a = H$ for all $a \in H$, while it is left closed if and only if $a\backslash a = H$ for all $a \in H$.*

Proposition 2. *The hypercomposition in a hypergroup H is right open if and only if $a/a = a$ for all $a \in H$, while it is left open if and only if $a\backslash a = a$ for all $a \in H$.*

Proposition 3. *If the hypercomposition in a hypergroup H is right or left open, then all its elements are idempotent.*

Two *induced hypercompositions* (the *left* and the *right division*) derive from the hypercomposition of the hypergroup [6], i.e.

$a/b = \{x \in H \mid a \in xb\}$ and $b\backslash a = \{y \in H \mid a \in by\}$

When "·" is commutative, $a/b = b\backslash a$. Consequences of the axioms (i) and (ii) are:

i. $ab \neq \varnothing$, for all $a,b \in H$.
ii. $a/b \neq \varnothing$ and $a\backslash b \neq \varnothing$, for all $a,b \in H$.
iii. the nonempty result of the induced hypercompositions is equivalent to the reproductive axiom.
iv. $(a/b)/c = a/(c{\cdot}b)$, $c\backslash(b\backslash a) = (b{\cdot}c)\backslash a$, $(b\backslash a)/c = b\backslash(a/c)$, for all $a,b,c \in H$ (mixed associativity) [7]

A *transposition hypergroupoid* is a hypergroupoid which satisfies the axiom [5]:

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing$$

A commutative transposition hypergroup is called *join hypergroup* or *join space* [5, 8].

In general *graph* is a set of points called *vertices* connected by lines called *edges*. A *path* in a graph is a sequence of no repeated vertices $v_1, v_2, \ldots, v_n$, such that $\overline{v_1v_2}, \overline{v_2v_3}, \ldots, \overline{v_{n-1}v_n}$, are edges in the graph. The *length* of a path is the number of edges that it uses. A graph is said to be *connected* if every pair of its vertices is connected by a path. A *directed graph* (or *digraph*) is a graph, where the edges have a direction associated with them. A degenerate edge of a graph which joins a vertex to itself, also called a *self-loop* or *loop*. *Multiple edges* are two or more edges that connect the same two vertices. The term *multigraph* refers to a graph which has multiple edges between nodes. A *directed graph* (or *digraph*) is a graph, where the edges have a direction associated with them. A *simple graph* (or *strict graph*), is an unweighted, undirected graph containing no graph loops or multiple edges. A *tree* $\mathscr{T}$ is a simple, connected graph with no

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cycles. A *spanning tree* of a connected graph is a tree whose vertex set is the same as the vertex set of the graph, and whose edge set is a subset of the edge set of the graph.

An *automaton* $\mathcal{A}$ is a collection of five objects (Σ, S, δ, s_o, F) where Σ is the *alphabet* of input letters (a finite nonempty set of symbols), S is a finite nonvoid set of *states*, s_o is the *start* (or *initial*) state, an element of S, F is the set of the *final* (or *accepting*) states, a (possibly empty) subset of S and δ is the *state transition function* with domain S×Σ and range S, in the case of a *deterministic* automaton (DFA), or $\mathcal{P}$(S), in the case of a *nondeterministic* automaton (NDFA). Σ^* denotes the set of *words* (or *strings*) formed by the letters of Σ –closure of Σ– and $\lambda\in\Sigma^*$ signifies the empty word. Σ^* under the concatenation of words is a monoid, with neutral element λ, since $\lambda x=x\lambda=x$ for all x in Σ^*. Moreover Σ^* becomes a hyperingoid under the b-hyperoperation: $x+y=\{x, y\}$ for all x, y in Σ^* [20]. Given a DFA $\mathcal{A}$, the *extended state transition function* for $\mathcal{A}$, denoted δ^*, is a function with domain S×Σ^* and range S defined recursively as follows:

i. $\delta^*(s,a) = \delta(s,a)$ for all s in S and a in Σ
ii. $\delta^*(s,\lambda) = s$ for all s in S
iii. $\delta^*(s,ax) = \delta^*(\delta(s,a),x)$ for all s in S, x in Σ^* and a in Σ.

P. Corsini [2], M. Gionfriddo [4], Nieminen [21, 22], I. Rosenberg [23], M. De Salvo et al. [24] and others studied hypergroups associated with graphs. G. G. Massouros [14-18] and after him J. Chvalina [1] studied hypergroups associated with automata. Moreover, in [15] G. G. Massouros introduced the path hypercomposition in graphs and subsequently Ch. G. Massouros and G. G. Massouros introduced in [9] another type of path hypercomposition in graphs and some relevant hypercompositions in automata.

2 The path hypercompositions in Graphs

In the set V of the vertices of a tree, a hypercompostion "·" has been introduced in [9] as follows: for each two vertices x, y in V, $x{\cdot}x=x$ and $x{\cdot}y$ is the set of all vertices which belong to the path that connects vertex x with vertex y. Since tree is an undirected graph, this hypercomposition is commutative. Furthermore, this hypercomposition is a closed hypercomposition. Therefore:

Proposition 4. *If V is the set of the vertices of a tree $\mathcal{T}$, then $V = x/x$, for each x in V.*

The set $<x,y> = x/y \cup x{\cdot}y \cup y/x$, where $x\neq y$ are two vertices of $\mathcal{T}$, is called the *line* of $\mathcal{T}$ which is defined by x, y. A subset S of V is called *convex*, if it holds $x{\cdot}y\subseteq S$, for each x, y in S. In [9] it is proved that the lines of $\mathcal{T}$ are convex sets. Moreover the following important theorem it is proved in [9]:

Theorem 1. *If V is the set of the vertices of a tree $\mathcal{T}$, then (V, ·) is a join space.*

It is known that any connected graph has at least one spanning tree and that there exist algorithms which find such trees. Hence any graph can be endowed with the join space structure through its spanning trees.

Theorem 2. *Let $\mathcal{G}$ be a connected graph and $\mathcal{T}$ a spanning tree of $\mathcal{G}$. The set of the vertices of the graph becomes a join space if for all vertices x, y of $\mathcal{G}$, the hypercomposition $x\bullet_{\mathcal{T}}y$ is the set of all vertices which belong to the path that connects vertex x with vertex y in $\mathcal{T}$.*

Since a graph may have more than one spanning trees, more than one join spaces can be associated to a graph.

Next, define in the set V of the vertices of a tree $\mathcal{T}$ a hypercomposition "•", such that for each two vertices x, y in V, $x\bullet y$ consists of all the internal vertices which belong to the path that connects vertex x with vertex y, that is, if $\overline{xv_1}, \overline{v_1v_2}, ..., \overline{v_ny}$ are edges in a path connecting the vertices x and y, then $x\bullet y=\{v_1,v_2,...,v_n\}$. This hypercomposition is an open hypercomposition. It is obvious that $(V,\bullet)$ is a partial hypergoupoid, since the result of the hypercomposition of two successive vertices is void. If the above hypercomposition is introduced in a simple connected graph, then it is possible to exist more than one paths connecting two vertices x, y of the graph. Hence if $\overline{xv_1}, \overline{v_1v_2}, ..., \overline{v_ny}$ are edges in a path which connects the vertices x and y, then $\{v_1,v_2,...,v_n\}\subseteq x\bullet y$. $(V,\bullet)$ will be a hypergroupoid if and only if for any two vertices x and y of V there exists a path from x to y of length greater or equal to 2.

The Boolean domain B = {0, 1} becomes a semiring under the addition

$$0 + 1 = 1 + 0 = 1 + 1 = 1,\ 0 + 0 = 0$$

and the multiplication

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0,\ 1 \cdot 1 = 1.$$

This semiring is called a binary Boolean semiring. A Boolean matrix is a matrix with entries from the binary Boolean semiring. A square Boolean matrix is called *total* if all its entries are equal to 1 [10]. The *adjacency matrix* of a graph on n vertices is an n x n Boolean matrix $A = (a_{i,j})$ in which the entry $a_{i,j}$ equals to 1, if there is an edge from vertex i to vertex j and equals to 0 if there is no edge from vertex i to vertex j. Through the adjacency matrix a binary relation ρ can be defined in in the set V of the vertices as follows:

$$(a_i,a_j)\in\rho \text{ if and only if } a_{ij}=1$$

In [11] the following theorem has been proved:

Theorem 3. *Let H be a non-empty set and ρ a binary relation on H. Then Corsini's hypercomposition in H:*

$$xy=\{z\in H \mid (x, z)\in\rho \text{ and } (z, y)\in\rho\}$$

endows H with the hypergroup structure if and only if $(x, y)\in\rho$, for all $x, y\in H$.

A consequence of the above theorem is the following theorem:

Theorem 4. *Let V be the set of the vertices of a graph $\mathcal{G}$. Then the hypercomposition in V:*

$$x \bullet y = \begin{cases} \{z \in H \mid x, z, y \text{ is a path in } \mathcal{G}\} & \text{if } x \neq y \\ x & \text{if } x = y \end{cases}$$

endows V with the hypergroup structure if and only if the adjacency matrix of $\mathcal{G}$ is total.

3 The path hypercompositions in Automata

In [14-19] it has been shown by G. Massouros, that the set of the states of an automaton, equipped with different hypercompositions, can be endowed with the structure of the hypergroup. The hypergroups that have derived in this way were named attached hypergroups to the automaton. Up to this point several kinds of attached hypergroups have introduced in order to describe the structure and the operation of the automata with the use of tools from the Hypercompositional Algebra. Among them there are:

i. the *attached hypergroups of the order*, and
ii. the *attached hypergroups of the grade*.

These two kinds of hypergroups have also been used for the minimization of the automata.

Moreover, in [15] another hypergroup, which derived through a different consideration of the hypercomposition, has been attached to the set of the states of an automaton. Due to its definition this hypergroup was named by G. Massouros *attached hypergroup of the paths* and it has led to a new proof of Kleene' s theorem. Furthermore, in [16], the *attached hypergroup of the operation* has been attached to the automaton. Apart from the other results, this hypergroup can indicated all the states in which an automaton can be found after the t - clock pulse.

Hereafter two hypercompositions will be presented which are defined through the strings of letters from the alphabet of the automaton. Let $\mathcal{A}$ be the automaton $(\Sigma, S, \delta, s_o, F)$. If x be a word in Σ^*, then:

$$\text{Prefix}(x)=\{y\in\Sigma^* \mid yz=x \text{ for some } z\in\Sigma^*\}$$

and $$\text{Suffix}(x)=\{z\in\Sigma^* \mid yz=x \text{ for some } y\in\Sigma^*\}$$

Let s be an element of S. Then

$$I_s = \{x\in\Sigma^* \mid \delta^*(s_o,x)=s\}$$

and $$P_s = \{s_i\in S \mid s_i=\delta^*(s_o,y),\ y\in\text{Prefix}(x),\ x\in I_s\}$$

Considering the automaton as a directed graph, P_s is the set of the states which appear in all possible paths connecting the start state s_o with the state s. Since $\lambda\in\Sigma^*$ the states s_o and s are in P_s. In the set of the states of $\mathcal{A}$ we introduce the hypercomposition

$$s + q = P_s \cup P_q \quad \text{for all } s, q \in S \qquad (1)$$

This hypercomposition is commutative, thus the two induced hypercompositions coincide and so we have:

$$s/q = q\backslash s = \begin{cases} S, & \text{if } s \in P_q \\ \{r \in S \mid P_s \subseteq P_r\} & \text{if } s \notin P_q \end{cases}$$

It is proved that [9]:

Proposition 5. *The set S endowed with the hypercomposition (1) is a join hypergroup.*

The other hypercomposition is defined as follows:

$$s + q = P_s \cap P_q \quad \text{for all } s, q \in S \qquad (2)$$

Since $s_o \in P_r$ for all $r \in S$, the results of hypercomposition (2) are always non void sets. Moreover this hypercomposition is commutative, thus the two induced hypercompositions coincide and so we have:

$$s/q = q\backslash s = \begin{cases} S, & \text{if } s \in P_q \\ \varnothing, & \text{if } s \notin P_q \end{cases}$$

It has been proved that [9]:

Proposition 6. *The set S endowed with the hypercomposition (2) is a join semihypergroup.*

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