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Global Stability of Complex-Valued Genetic Regulatory Networks with Delays on Time Scales

Yajing Wang[1], Zhenkun Huang[1] , Juan Chen[1]

[1] *Jimei University, Xiamen 361021, China*

Abstract. In this paper, the global exponential stability of complex-valued genetic regulatory networks with delays is investigated. Besides presenting conditions guaranteeing the existence of a unique equilibrium pattern, its global exponential stability is discussed. Some numerical examples for different time scales.

Keywords: Complex-Valued Genetic Regulatory Networks, Time Scales, Delay, Global Exponential Stability, - globally exponentially stable.

1 Introduction

The study of complex-valued neural networks (CVNNs for short) is a fast growing area of research in recent times as is apparent from a large number of publications. In fact, complex-valued neural networks (CVNN) make it possible to solve some problems which cannot be solved with their real-valued counterparts. Stimulated by works in [1,2], We consider the generalized CVNN described by the equation.

$$\begin{cases} z^{\Delta}(t) = -Az(t) + Bf(\hat{z}(t-\tau)) + L, \\ \hat{z}^{\Delta}(t) = -P\hat{z}(t) + Dz(t-\tau), \end{cases} \tag{1}$$

where T be a time scale and C be the set of complex numbers, $z: T \to C^n, \hat{z}: T \to C^n, A, B, P$ and D are $n \times n$ -matrices with complex entries, the activation functions are given by $f: C^n \to C^n$, and the inputs are given by $L \in C^n$.

2 Essentials of time scales

An arbitrary nonempty closed subset T of the set of real numbers R is called a time scale. In this paper, we only consider time scales that are unbounded above. We define the forward jump operator $\sigma: T \to T$ (and similarly the backward jump operator ρ) by $\sigma(t) = \inf\{s \in T : s > t\}$. A point t $t \in T$ is called right-scattered, right-dense, left-scattered, left-dense, if $\sigma(t) > t, \sigma(t) = t, \rho(t) > t, \rho(t) = t$ holds, respectively. The graininess $\mu: T \to [0]$ is defined by $\mu(t) = \sigma(t) - t$. For $f: T \to C^n$ and $t \in T$, we note that the real and imaginary parts of f are real valued and one can use the time scales results below for the real-valued entries of $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. We say that $f: T \to R$ is delta differentiable at $t \in T$ provided there exists an α such that for all $\varepsilon > 0$ there is a neighborhood $\mathcal{N}$ of t with $|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|$ for all $s \in \mathcal{N}$.

It is easy to see that

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{s \to t, s \in T} \frac{f(t) - f(s)}{t - s}, & \mu(t) = 0, \\ \frac{f(\sigma(t)) - f(t)}{\mu(t)}, & \mu(t) \succ 0. \end{cases}$$

and

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \tag{2}$$

where we put $f^{\sigma} = f \circ \sigma$, and the simple useful formula

$$f^{\sigma} = f + \mu f^{\Delta}. \tag{3}$$

We say that a function $f: T \to R$ is regressive provided $1 + \mu(t) f(t) \ne 0$ for all $t \in T$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R}$. The set $\mathcal{R}^+$ of all positively regressive function consists of those $f \in \mathcal{R}$ that satisfy $1 + \mu(t) f(t) > 0$ for all $f \in \mathcal{R}$.

Lemma 1 Let $p, q \in \mathcal{R}$ and $t, s, r \in T$. Then

(i) $e_0(t, t_0) \equiv 1$ and $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t, s)$;

(iii) if $T = R$ and $p(t) \equiv \alpha$, then $e_p(t, s) = e^{\alpha(t-s)}$;

(iv) if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t > s$.

Lemma 2 Let $y \in C_{rd}$ and $p \in \mathcal{R}^+$. If y is differentiable on $[t_0,\infty)\cap T$ such that

$$y^{\Delta}(t) \le p(t)y(t) \quad \text{for all } t\in[t_0,\infty)\cap T,$$

then

$$y(t) \le y(t_0)e_p(t,t_0) \quad \text{for all } t\in[t_0,\infty)\cap T.$$

3 Existence of a unique equilibrium pattern

Consider the space C^n of all *n*-vectors of complex numbers and let $|z| = \sqrt{z^* z}$ denote the absolute value of $z \in C^n$, where * indicates the conjugate transpose.

Theorem 3 Suppose $f: C^n \to C^n$ is continuous with $f(0)=0$ and there exists $L>0$ such that $|f(z_1)-f(z_2)| \le L|z_1-z_2|$ for all $z_1, z_2 \in C^n$. Define

$$\gamma := \|I-A\| + \|B\|\|P^{-1}D\|L.$$

If P is invertable and $\gamma \in (0,1)$, then the system (1) possesses a unique equilibrium pattern.

Proof Clearly, the assumptions of the theorem imply $|f(z)| \le L|z|$ for all $z \in C^n$. Now define the operator $F: C^n \to C^n$ by

$$F(z) = (I-A)z + Bf(P^{-1}Dz) + L.$$

For $z_1, z_2 \in C^n$, we have

$$|F(z_1)-F(z_2)| = \left|(I-A)(z_1-z_2) + B\left[f(P^{-1}Dz_1) - f(P^{-1}Dz_2)\right]\right| \le \gamma|z_1-z_2|.$$

Thus the mapping F is a contraction on C^n. Therefore, F has a unique fixed point z in C^n since C^n is complete. Let $\hat{z}^0 = P^{-1}Dz^0$, then we get

$$\begin{cases} -C\hat{z}^0 + Dz^0 = 0, \\ -Az^0 + Bf(\hat{z}^0) + L = 0. \end{cases}$$

This unique fixed point $(z^0, \hat{z}^0)$ is the required equilibrium pattern for the system (1).

4 Global exponential stability

We assume that the system (1) possesses a unique equilibrium pattern $(\dot{z}, \ddot{z})^T$. Using the transformation $\overline{z}(t) = z(t) - \dot{z}, \overline{\overline{z}}(t) = \hat{z}(t) - \ddot{z}$ in system (1), we get

$$\begin{cases} \overline{z}^{\Delta}(t) = -A\overline{z}(t) + Bg(\overline{\overline{z}}(t-\tau)), \\ \overline{\overline{z}}^{\Delta}(t) = -P\overline{\overline{z}}(t) + D\overline{z}(t-\tau). \end{cases}$$

where $g(z) = f(z+\ddot{z}) - f(\ddot{z})$. Redesignating $(\overline{z}(t), \overline{\overline{z}}(t))^T$ as $(z(t), \hat{z}(t))^T$ we obtain

$$\begin{cases} z^{\Delta}(t) = -Az(t) + Bg(\hat{z}(t-\tau)), \\ \hat{z}^{\Delta}(t) = -P\hat{z}(t) + Dz(t-\tau). \end{cases} \tag{4}$$

Clearly the stability of $(\dot{z}, \ddot{z})^T$ for the system (1) is equivalent to the stability of the trivial solution for the system (4). We use the following concept of global exponential stability.

Lemma 6 If $(z, \hat{z})^T$ is a solution of (4), then $w = z^*z + \hat{z}^*\hat{z}$ and $w_1 = z^*z$, $w_2 = \hat{z}^*\hat{z}$ satisfies

$$\begin{aligned} w^{\Delta} &= z^*(-A^* - A + \mu A^*A)z + \mu g^*(\hat{z}(t-\tau))B^*Bg(\hat{z}(t-\tau)) \\ &+ g^*(\hat{z}(t-\tau))B^*(I-\mu A)z + z^*(I-\mu A)^*Bg(\hat{z}(t-\tau)) \\ &+ \hat{z}^*(-P^* - P + \mu P^*P)\hat{z} + \mu z^*(t-\tau)D^*Dz(t-\tau) \\ &+ \hat{z}^*(I-\mu P)^*Dz(t-\tau) + z^*(t-\tau)D^*(I-\mu P)\hat{z}. \end{aligned}$$

Proof We use the product rule (2) and the simple useful formula (3) to calculate

$$\begin{aligned} w^{\Delta} &= (-z^*A^* + g^*(\hat{z}(t-\tau))B^*)z + z^*(-Az + Bg(\hat{z}(t-\tau))) \\ &+ \mu(-z^*A^* + g^*(\hat{z}(t-\tau))B^*)(-Az + Bg(\hat{z}(t-\tau))) \\ &+ (-\hat{z}^*P^* + z^*(t-\tau)D^*)\hat{z} + \hat{z}^*(-P\hat{z} + Dz(t-\tau)) \\ &+ \mu(-z^*P^* + z^*(t-\tau)D^*)(-P\hat{z} + Dz(t-\tau)) \\ &= z^*(-A^* - A + \mu A^*A)z + \mu g^*(\hat{z}(t-\tau))B^*Bg(\hat{z}(t-\tau)) \\ &+ g^*(\hat{z}(t-\tau))B^*(I-\mu A)z + z^*(I-\mu A)^*Bg(\hat{z}(t-\tau)) \\ &+ \hat{z}^*(-P^* - P + \mu P^*P)\hat{z} + \mu z^*(t-\tau)D^*Dz(t-\tau) \\ &+ \hat{z}^*(I-\mu P)^*Dz(t-\tau) + z^*(t-\tau)D^*(I-\mu P)\hat{z}. \end{aligned}$$

This completes the proof.

Lemma 7 Let $\omega > 0$, $\upsilon > 0$. If $(z, \hat{z})^T$ is a solution of (4), then $w = z^*z + \hat{z}^*\hat{z}$ satisfies

$$\begin{aligned} w^{\Delta} &\le z^*(-A^* - A + \mu A^*A + \frac{1}{\omega}(I-\mu A)^*(I-\mu A))z \\ &+ (\mu+\omega)g^*(\hat{z}(t-\tau))B^*Bg(\hat{z}(t-\tau)) \\ &+ \hat{z}^*(-P^* - P + \mu P^*P + \frac{1}{\upsilon}(I-\mu P)^*(I-\mu P))\hat{z} \\ &+ (\mu+\upsilon)z^*(t-\tau)D^*Dz(t-\tau). \end{aligned}$$

Proof First notice that for $\omega > 0, \upsilon > 0$ we have

$$\begin{aligned} 0 &\le (\sqrt{\omega}Bg(\hat{z}(t-\tau)) - \frac{1}{\sqrt{\omega}}(I-\mu A)z)^* \\ &\quad (\sqrt{\omega}Bg(\hat{z}(t-\tau)) - \frac{1}{\sqrt{\omega}}(I-\mu A)z) \\ &= \omega g^*(\hat{z}(t-\tau))B^*Bg(\hat{z}(t-\tau)) - g^*(\hat{z}(t-\tau))B^*(I-\mu A)z \\ &- z^*(I-\mu A)^*(I-\mu A)g(\hat{z}(t-\tau)) \\ &+ \frac{1}{\omega}z^*(I-\mu A)^*(I-\mu A)z, \end{aligned}$$

$$\begin{aligned} 0 &\le (\sqrt{\upsilon}Dz(t-\tau) - \frac{1}{\sqrt{\nu}}(I-\mu P)\hat{z})^* \\ &\quad (\sqrt{\upsilon}Dz(t-\tau) - \frac{1}{\sqrt{\nu}}(I-\mu P)\hat{z}) \\ &= \upsilon z^*(t-\tau)D^*Dz(t-\tau) - z^*(t-\tau)D^*(I-\mu P)\hat{z} \end{aligned}$$

$$-\hat{z}^*(I-\mu P)^* Dz(t-\tau)+\frac{1}{\upsilon}\hat{z}^*(I-\mu P)^*(I-\mu P)\hat{z}.$$

Using this in Lemma 6, we can complete the proof.

Theorem 8 Suppose g satisfies a Lipschitz condition with Lipschitz constant L. Assume that $A=diag(a_1,a_2,\ldots,a_n), P=diag(p_1,p_2,\ldots,p_n)$ are complex-valued diagonal matrix. If there exist $\omega>0, \upsilon>0$ such that

$$\phi:=\max\left\{\tilde{a}+(\mu+\upsilon)\lambda_2, \tilde{p}+(\mu+\omega)L^2\lambda_1\right\},$$

satisfies $\phi\in\mathcal{R}^+$ and $\lim_{t\to\infty} e_\phi(t,t_0)=0,$ where λ_1 is the maximal eigenvalue of B^*B, λ_2 is the maximal eigenvalue of $D^*D,$ and

$$\tilde{a}:=\max_{1\le i\le n}\left\{-2\operatorname{Re}(a_i)+\mu|a_i|^2+\frac{1}{\omega}|1-\mu a_i|^2\right\},$$

$$\tilde{p}:=\max_{1\le i\le n}\left\{-2\operatorname{Re}(p_i)+\mu|p_i|^2+\frac{1}{\upsilon}|1-\mu p_i|^2\right\},$$

then the trivial solution of (4) is ϕ -globally exponentially stable.

Proof Let $(z,\hat{z})^T$ be any solution of (4) and let $w=w_1+w_2,\ w_1=z^*z,\ w_2=\hat{z}^*\hat{z}.$ Then

$$g^*(\hat{z}(t-\tau))B^*Bg(\hat{z}(t-\tau))\le L^2\lambda_1 w_2,$$

$$z^*(t-\tau)D^*Dz(t-\tau)\le\lambda_2 w_1,$$

and

$$z^*(-A^*-A+\mu A^*A+\frac{1}{\omega}(I-\mu A)^*(I-\mu A))z\le\tilde{a}w_1,$$

$$\hat{z}^*(-P^*-P+\mu P^*P+\frac{1}{\upsilon}(I-\mu P)^*(I-\mu P))\hat{z}\le\tilde{c}w_2.$$

Thus ,by Lemma 7, we have

$$w^\Delta\le\tilde{a}w_1+(\mu+\omega)L^2\lambda_1 w_2+\tilde{p}w_2+(\mu+\upsilon)\lambda_2 w_1\le\phi(w_1+w_2)=\phi w.$$

Now Lemma 2 yields

$$w(t)\le w(t_0)e_\phi(t,t_0)\to 0 \quad \text{as} \quad t\to\infty,$$

so that the claim follows.

5 Examples

Consider the network

$$\begin{cases} z^\Delta(t)=-\begin{pmatrix}0.99 & 0\\ 0 & 0.988\end{pmatrix}z(t)+\begin{pmatrix}0.025+0.025i & -0.05+0.025i\\ 0.075-0.05i & -0.025+0.025i\end{pmatrix}g(\hat{z}(t-\tau)),\\ \hat{z}^\Delta(t)=-\begin{pmatrix}0.99 & 0\\ 0 & 0.995\end{pmatrix}\hat{z}(t)+\begin{pmatrix}0.035+0.035i & -0.05+0.035i\\ 0.065-0.05i & -0.035+0.035i\end{pmatrix}z(t-\tau).\end{cases} \quad (5)$$

In (5), we choose the function

$$g(x)=\begin{pmatrix}\tanh(x_1+x_2)\\ \tanh(x_1+x_2)\end{pmatrix},$$

the Lipschitz constant is equal to 1 and $\omega=\upsilon=1,$ so that $\tilde{a}=-0.999712, \tilde{p}=-0.9998.$ The two eigenvalues of B^*B can be computed as 0.0120 and $0.0017.$ The two eigenvalues of D^*D can be computed as 0.0017 and 0.0136 so that $\lambda_1=0.0120, \lambda_2=0.0136.$ Hence

$$\phi=\max\{-0.999712+(\mu+\omega)0.0136, -0.9998+(\mu+\upsilon)0.0120\},$$

is constant if μ,ω and υ are constant , in which case we write $\alpha=\phi.$

1.For $T=R$ we have $\mu(t)\equiv 0.$ We find $\alpha=-0.986112$ and thus $e_\alpha(t,0)=e^{\alpha t}\to 0$ *as* $t\to\infty$ so that the trivial solution of (5) is α -globally exponentially stable.

2.For $T=Z$ we have $\mu(t)\equiv 1.$ We find $\alpha=-0.7598$ and thus $e_\alpha(t,0)=(1+\alpha)^t\to 0$ as $t\to\infty$ so that the trivial solution of (5) is α -globally exponentially stable.

6 Discussion

In this paper we have studied complex-valued genetic regulatory networks with delays on time scales. Sufficient conditions for the existence of a unique equilibrium solution are derived. The global exponential stability conditions derived in this paper are new, fairly general and offer greater flexibility in handing time scales of practical importance.

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