MATEC Web of Conferences 31, 17006 (2015)
DOI: 10.1051/matecconf/20153117006

The research of (2,1)-total labelling of trees basen on Frequency Channel Assignment problem

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Abstract. Let T be a tree, Let $D_\Delta(T)$ denote the set of integers k for which there exist two distinct vertices of maximum degree of distance at k in T. The (2,1)-total labelling number of a graph G is the width of the smallest range of integers that suffices to label the vertices and the edges of G such that no two adjacent vertices have the same label, no two adjacent edges have the same label and the difference between the labels of a vertex and its incident edges is at least 2. In this paper, we prove that if T is a tree with $\Delta \geq 5$ and $3,4 \notin D_\Delta(T)$, then T is Type 1.

1 Introduction

Motivated by the Frequency Channel Assignment problem, Griggs and Yeh[1]introduced the $L(2,1)$-labelling of graphs. This notion was subsequently generalized to the $L(p,q)$-labelling problem of graphs. Let p and q be two nonnegative integers. An $L(p,q)$-*labelling* of a graph G is a function f from its vertex set $V(G)$ to the set $\{0,1,\cdots,k\}$ such that $|f(x)-f(y)|\geq p$ if x and y are adjacent, and $|f(x)- f(y)|\geq q$ if x and y are at distance 2. The $L(p,q)$ *-labelling number* $\lambda_{p,q}(G)$ of G is the smallest k such that G has an $L(p,q)$-labelling f with max $\{f(v)v\in V(G)\}=k$ .In particular, we simply write $\lambda(G)=\lambda_{2,1}(G)$. The $L(p,q)$ -labelling of graphs have been studied rather extensively in recent years [2,3,4,5,6]. Whittlesey, Georges and Mauro [7] investigated the $L(2,1)$-labelling of incidence graphs. The *incidence graph* of a graph G is the graph obtained from G by replacing each edge by a path of length 2. The $L(2,1)$-labelling of the incidence graph of G is equivalent to an assignment of integers to each element of $V(G)\bigcup E(G)$ such that adjacent vertices have different labels, adjacent edges have different labels, and incident vertex and edge have the difference of labels by at least 2. This labelling is called $(2,1)$-total labelling of graphs, which was introduced by Havet and Yu [8] and generalized to $(d,1)$-total labelling. Let $d\geq 1$ be an integer. A k-$(d,1)$ *-total labelling* of a graph G is a function f from $V(G)\bigcup E(G)$ to the set $\{0,1,\cdots,k\}$ such that $f(u)\neq f(v)$ if u and v are two adjacent vertices, $f(e)\neq f(e')$ if e and e' are two adjacent edges, and $|f(u)-f(e)|\geq d$ if a vertex u is incident to an edge e.

Let $\Delta(G)$ (or simply Δ) denote the maximum degree of a graph G. Suppose that T is a tree with $\Delta\geq 3$. It is easy to prove that $\Delta+1\leq\lambda(T)\leq\Delta+2$ and $\Delta+1\leq\lambda_2^t(T)\leq\Delta+2$. Wang [9] showed that if T does not contain two vertices of maximum degree at distance either 1, 2, or 4, then $\lambda(T)=\Delta+1$. Moreover, examples of trees T having two vertices of maximum degree at distance 4 such that $\lambda(T)=\Delta+2$ was constructed in [10].

A star is a tree that consists of Δ leaves (A leaf is a $1-$vertex) and a $\Delta-$vertex. A generalized star is a tree that all vertices are leaves except that two adjacent vertices.A star is also a generalized star , the $L(2,1)-$total labelling number of a generalized star is $\Delta+1$. A tree T is called Type 1 if $\lambda_2^t(T)=\Delta+1$ and Type 2 if $\lambda_2^t(T)=\Delta+2$.Let M denote a generalized star, a tree of order 8 consisting two adjacent 4-vertices and four leaves. Let $K_{1,4}$ denote the star of order 5. Clearly, both $K_{1,4}$ and M are Type 1.

Let T be a tree with $\Delta\geq 3$. We use $|T|$ to denote the number of vertices of T, and $d(v)$ to denote the degree of a vertex v in T. A vertex of degree k is called a k-*vertex*. The vertex v is called *major* if $d(v)=\Delta$, *minor*

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if $d(v) < \Delta$, a *leaf* if $d(v) = 1$, and a *handle* if $d(v) > 1$ and v is adjacent to at most one vertex of degree greater than one. A k *-handle* is a handle of degree k. A *major handle* is a vertex of maxi- mum degree that is a handle. Obviously, every tree T with $|T| \geq 3$ that is not a star contains at least two handles. Let $\ell(P)$ denote the length of a path P. The *distance* between two vertices x and y in T is the length of a shortest path connecting them. The *diameter*, denoted by $\mathrm{diam}(T)$, of T is the maximum distance between two vertices of T. Let $D_\Delta(T)$ denote the set of integers k for which there exist two distinct major vertices of distance at k in T.

A *generalized star* is a tree T with at least $|T|-2$ leaves. Obviously, the star $K_{1,n}$ is a special generalized star, which has $n = |K_{1,n}|-1$ leaves. For a vertex $u \in V(T)$, we use $L(u)$ to denote the set of leaves which adjacent to u. For $S \subseteq V(G)$, let $G-S$ denote the graph obtained from G by deleting all the vertices in S. Also a tree T is called Type 1 if $\lambda_3^t(T) = \Delta+3$ and Type 2 if $\lambda_3^t(T) = \Delta+4$.

Given an edge $e = vu \in E(T)$, we use $T_v(e)$ to represent the subtree of T which is rooted at the vertex v and contains the edge e.

Lemma 1 [11]**.** Let T be a tree with $\Delta = 4$ that is not a generalized star. If $3,4 \notin D_\Delta(T)$, then T contains one of the following configurations:

(C1) A leaf u is adjacent to a minor vertex v.

(C2) A path $x_1x_2x_3x_4$ such that $d(x_2) = 2$ and x_1 is a major handle.

(C3) A path $x_1x_2x_3x_4$ such that $d(x_2) = 3$, x_1 is a major handle and $x_2y_1y_2$ is a path, where y_1 is a neighbor of x_2 with y_1 is a major handle. $deg(x_3) = 2$ or 4.

(C4) A path $x_1x_2x_3x_4$ such that x_1 is a major handle and x_2 is a weak major handle.

(C5) A path $x_1x_2x_3x_4$ such that x_1 is a major handle and x_2 is $\Delta-$ degree, $x_2y_1y_2$ is a path, where y_1 is a neighbor of x_2 with y_1 is a major handle, the neighbor of x_2 other than x_1, x_3 and y_1 is a leaf.

Lemma 2 [11]**.** Let T be a tree with $\Delta \geq 5$ that is not a star. If 3, 4 $\notin D_\Delta(T)$, then T contains one of the following configurations:

(B1) A leaf is adjacent to a minor vertex.

(B2) A minor vertex v is adjacent to $d(v)-1$ major handles.

(B3) A major handle is adjacent to a weak major handle.

(B4) A path $x_1x_2x_3x_4$ such that x_1 is a major handle and x_2 is Δ-degree, x_2 adjacent to $\Delta-2$ major handles at most.

2 Conclusion

Theorem 3. If T is a tree with Δ≥5 and $3,4 \notin D_\Delta$, then T is Type 1.

Proof The proof is proceeded by induction on $|T|$. The theorem holds clearly if $|T| = 5$. Let T be a tree with $|T| \geq 6$, $\Delta \geq 5$ and $3,4 \notin D_\Delta(T)$. If T is a generalized star, it is easy to construct a $(2,1)$-total labelling of T using the label set $B = \{0,1,\cdots,\Delta+1\}$. Thus, assume that T is not a generalized star. We need to diverse the following four cases.

If T contains (B1), then T contains a leaf x_1 adjacent to a minor vertex x_2, then $T - x_1$ has a (2,1)-total labelling f using $B = \{0,1,\cdots,\Delta+1\}$ by the induction hypothesis. Since $d(x_2) \leq \Delta-1$, there exist at most $\Delta-2+3 = \Delta+1$ forbidden labels for the edge $x_1\ x_2$ and at most four forbidden labels for the vertex x_1. By $|B| = \Delta+2$, we can first extend f to $x_1\ x_2$ and then to x_1. Hence, assume that no leaf is adjacent to a minor vertex.

If T contains (B2), then T contains a configuration: ,a minor vertex a adjacent to $d(a)-1$ major handles $x_1, x_2, \cdots, x_{d(a)-1}$ and the other vertex *b*. Let

$$T' = T - \bigcup_{i=1}^{d(a)-1} (L(x_i) \cup \{x_i\}).$$

By the induction hypothesis, T' has a $(2,1)$-total labelling f using $B = \{0,\cdots,\Delta+1\}$.

If $f(a) \notin \{0,\Delta+1\}$, then f can be extended into a $(2,1)$-total labelling of T. Assume that $f(a) \in \{0,\Delta+1\}$. Relabel a with a label from.

$$B - \{0, \Delta+1, f(b), f(ab)-1, f(ab), f(ab)+1\}$$

Since

$$|B - \{0, \Delta+1, f(b), f(ab)-1, f(ab), f(ab)+1\}|$$
$\geq |B|-6 = \Delta+2-6 \geq 5+2-6 = 1$, such a relabelling is feasible, f can be extended to T.

If T contains (B3): A major handle is adjacent to a weak major handle. We let x_1 be a major handle and x_2 a weak major handle. If $f(x_2)$ =0, we relable x_1 , x_1 x_2 with 5,3. If $f(x_2)$ =5, we relable x_1 , x_1 x_2 with 0,2. Because the other adjacent vertices of x_1 are leaves , we can easily conclude that T is Type 1.

If T contains (B4) : A path $x_1x_2x_3x_4$ such that x_1 is a major handle and x_2 is Δ degree, x_2 adjacent to $\Delta-2$ major handles at most. We let

$$T' = T - \{x_1\} - L(x_1) + x_2 + a$$

Where *a* is a leaf. By the induction hypothesis, T' has a (2,1)-total labelling . Because in T' , $f(x_2)$=0 or 5, so we diverse the following two cases:

If $f(x_2)$=0, then we exchange *a* with x_1, we label x_1 x_2 with label in *B*- {0,1, $f(x_2u_1)$, $f(x_2u_2)$,---, $f(x_2u_{\Delta-2})$ },Where u_1 , $u_{2,}$ ---, $u_{\Delta-2}$ are the adjacent vertices of x_2 and different from x_1 and x_3 . Since

$$\left| B\text{-}\{0, 1, f(x_2u_1), f(x_2u_2), \text{---}, f(x_2u_{\Delta-2})\}\right| \geq \Delta+1-\Delta=1,$$

so this label is feasible. Then we label x_1 with 5.

If $f(x_2)$ =5, we use the same method. We exchange *a* with x_1, we label x_1 x_2 with label in *B*- {0,1, $f(x_2u_1)$, $f(x_2u_2)$,---, $f(x_2u_{\Delta-2})$ },Where u_1 , $u_{2,}$ ---, $u_{\Delta-2}$ are the adjacent vertices of x_2 and different from x_1 and x_3 . Since

$$\left| B\text{-}\{0,1, f(x_2u_1), f(x_2u_2), \text{---}, f(x_2u_{\Delta-2})\}\right| \geq \Delta+1-\Delta=1,$$

so this label is feasible. Then we label x_1 with 0.

3 ACKNOWLEDGEMENTS

This article is supported by Natural Science Foundation of Ningbo City (under) Grant 2013A610067.

REFERENCES

1. W. K. Hale: Frequency assignment: theory and applications, Proc IEEE 68(1980), p.1497-1514.
2. J. R. Griggs and R. K. Yeh: Labelling graphs with a condition at distance 2, SIAM J Discrete Math 5(1992), p.586-595.
3. G. J. Chang and D. Kuo: The labelling problem on graphs, SIAM J Discrete Math 9(1996), p. 309-316.
4. G. J. Chang, W.-T. Ke, D. Kuo, Daphne D.-F. Liu, and R. K. Yeh: On labelling of graphs, Discrete Math 220(2000), p. 57-66.
5. J. P. Georges, D. W. Mauro, and M. I. Stein: Labeling products of complete graphs with a condition at distance two, SIAM J Discrete Math 14(2000), p. 28-35.
6. D. Kr′al and R. ˇSkrekovski: A theorem about the channel assignment problem, SIAM J Discrete Math 16(2003), p.426-437.
7. M. Molloy and M. R. Salavatipour: A bound on the chromatic number of the square of a planar graph, J Combin Theory Ser B 94(2005), p.189-213.
8. D. Sakai: Labelling chordal graphs: distance two condition, SIAM J Discrete Math 7(1994), p.133-140.
9. F.Bazzaroa,M. Montassier, and A. Raspaud: total labelling of planar graphs with large girth and high maximum degree, Discrete Math, in press.
10. D. Chen and W. Wang, total labelling of outerplanar graphs, submitted.
11. Haina Sun, Trees with 3,4 are not in maximum degree distance, Lecture Notes in Information Technology, 3(2011),p. 91-95