MATEC Web of Conferences **16**, 06001 (2014)
DOI: 10.1051/matecconf/20141606001

Regularization method for calibrated POD reduced-order models

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Abstract. In this work we present a regularization method to improve the accuracy of reduced-order models based on Proper Orthogonal Decomposition. The bench mark configuration retained corresponds to a case of relatively simple dynamics: a two-dimensional flow around a cylinder for a Reynolds number of 200. Finally, we show for this flow configuration that this procedure is efficient in term of reduction of errors.

1 Introduction

The numerical simulations for complex flows, such as solving the governing Navier-Stockes equations describing three dimentional flows with high Reynolds numbers, generate a high computational costs and a loss of accuracy of the obtained numerical solutions. This reality has motivated efforts to develop Reduced Order Models (ROM) by transfering the model of high number of degrees of freedom (DOF), either from detailed numerical simulation (DNS) or experimental measurements, with a fewer number of DOF. The most existing approaches are based on the projection of the high dimentional space into a low dimensional subspace. In this work, we use the ROM method obtained by Galerkin Projection (GP) of the governing equation onto an optimal set of a basis functions and representing the state space of the model using olny the first few basis functions containing the highest energy. This optimal basis, called POD modes, is constructed using the Proper Orthogonal Decomposition (POD).

In most of the applications, the reduced order models obtained may not be accurate due to the gap between the data used and the numerical implementation of the variational formulation assumed in the Galerkin projection, the truncation of the POD basis by keeping only the main POD modes, and others approximations. It's thus necessary to develop a practical methods to improve the accuracy, or in other word to calibrate, the reduced order models. Recently, various studies [2][3][4][5][6][8], presenting numerical methods termed as calibration, appeared in the literature in order to improve the accuracy of POD-based reduced-order models thanks to solutions of optimization problems. We find in [1] a comparative study of these calibration methods. The idea is simple, since the temporal dynamics of the POD model is known in advance, it is possible to use this information to correct whole or part of the coefficients issued from POD Galerkin.

This paper is organized as follow. We first explain how to construct a reduced order-model based on proper orthogonal decomposition. Then, we present the proposed approach based on regularization technique. Using a benchmark configuration, we analyse and demonstrate the efficiency of our approach.

2 POD Reduced-Order Model

Given a set $\mathcal{U} = \{\mathbf{u}(\mathbf{x}, t_m) = \mathbf{u}^m\}_{m=1,\ldots,N_t}$ of N_t snapshots obtained by numerical similations or experimentally. the POD method provides N_t mutually orthogonal basic functions, or spatial modes $\phi_i(x)$, which are optimal with respect to average kenetic energy representation of the flux. However, when the input data come from numerical simulations, it is much more efficient to use an alternate way of computing the POD eigenfunctions. The Sirovich's snapshots method consists of writing the POD modes as linear combinations of the snapshots, we obtain another eigenvalue problem, whose eigenfuctions are the temporal modes a_k. For example, the velocity over the POD modes ϕ_j writes:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_m(\mathbf{x}) + \sum_{j=1}^{N_{\text{POD}}} a_j(t)\mathbf{\Phi}_j(\mathbf{x}), \quad \text{with} \quad \mathbf{u}_m = E(u) \tag{1}$$

The POD Reduced-Order Model (POD ROM) is then constructed by applying the Galerkin projection to the governing equations into the subspace generated by a reduced number N_{gal} of POD modes. The Galerkin projection is given by,

$$(\partial_t \mathbf{u}, \mathbf{\Phi}_i)_\Omega = (\mathcal{N}(\mathbf{u}), \mathbf{\Phi}_i)_\Omega - (\nabla p, \mathbf{\Phi}_i)_\Omega \tag{2}$$

with

$$\mathcal{N}(\mathbf{u}) = -(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \frac{1}{Re}\Delta\mathbf{u}. \tag{3}$$

By substituting (1) into the Galerkin projection (3), we obtain after some algebraic manipulations the following expression for the reduced-order model without control:

$$\begin{cases} \dot{a}_i(t) = A_i + B_{ij}\,a_j(t) + C_{ijk}\,a_j(t)a_k(t) - P_i(t), \\ a_i(0) = a_i^{POD}(0) = (\mathbf{u}(\mathbf{x}, 0) - \mathbf{u}_m(\mathbf{x}), \mathbf{\Phi}_i)_\Omega\,. \end{cases}$$

where the Einstein summation is used and all subscripts i, j, k run from 1 to N_{gal}. Here, N_{gal} corresponds to the

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number of Galerkin modes retained in the reduced-order model. This number of modes is assumed to be sufficient to reproduce accurately the flow. The coefficients A_i, B_{ij} and C_{ijk} depend explicitly on $\mathbf{\Phi}$, $\mathbf{u}_m$ and $\mathbf{u}_c$. Whole or part of the coefficients may either be unknown or known with an insufficient level of accuracy to reproduce correctly the original dynamics. We can consider without restricting the generality $P_i(t) = 0$.

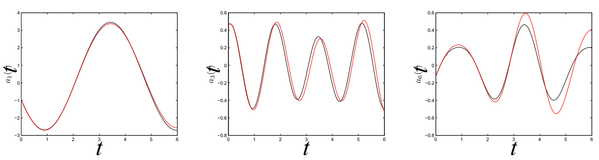
In vectorial formulation, the controlled POD ROM is finaly:

$$\begin{cases} \dot{\mathbf{a}}(t) = \mathbf{f}(\mathbf{y}, \mathbf{a}(t)), \\ \mathbf{a}(0) = \mathbf{a}^{POD}(0). \end{cases} \tag{4}$$

where

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N_{\text{gal}}} \end{pmatrix} \quad ; \quad \mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_{N_{\text{gal}}} \end{pmatrix} \quad \text{with} \quad \mathbf{y}_i = \begin{pmatrix} A_i \\ B_{i1} \\ \vdots \\ B_{iN_{\text{gal}}} \\ Q_{i11} \\ \vdots \\ Q_{iN_{\text{gal}}N_{\text{gal}}} \end{pmatrix}$$

The reduced-order modeling approach based on POD is now applied to a two-dimensional incompressible cylinder wake flow at $Re = 200$. The database was computed using a finite-element code (DNS code Icare, IMFT/University of Toulouse) and contains $N_t = 200$ two-dimensional snapshots of the flow velocity, taken over a period $T = 12$ *i.e.* over more than two periods of vortex shedding ($T_{\text{vs}} = 5$). The method of snapshots is applied to the velocity fluctuation. As shown in Fig. 2, the energy is concentrated in a very small number of modes: the first six POD modes are sufficient to represent 99.9% of the flow energy and we thus consider $N_{\text{gal}} = 6$ to derive the POD ROM. The POD ROM (4) is then integrated in time with a classical fourth-order Runge-Kutta scheme and a time step of $10^{-3}T$. A set of predicted time histories for the mode amplitudes $a_i(t)$ is obtained, and compared to the set of POD temporal eigenfunctions $a_i^{POD}(t)$. As shown in Fig. 1, the original dynamics is globally well reproduced but the accuracy is not perfect. It is thus necessary to introduce stabilization techniques in order to reproduce accurately the dynamics of reference using the POD ROM.



(a) Mode 1 (b) Mode 3 (c) Mode 6

Fig. 1. Comparison between the temporal evolutions of the projected modes and the predicted modes obtained by the initial system.

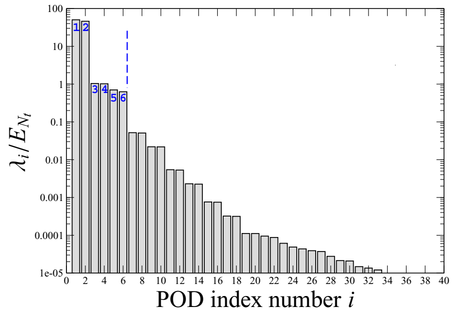


Fig. 2. POD eigenvalues in logarithmic scale. E_{N_t} corresponds to twice the energy contained in the database ($E_{N_t} = \sum_{j=1}^{N_t} \lambda_j$).

3 Proposed approach

Many methods of calibration have already been proposed in the literature. Let us quote for example the different techniques based on least-square minimization [2,3,5], those consisting in solving a constrained optimization problem, iteratively [7] or simultaneously [6], the recent method termed intrinsic stabilization introduced in [8], and finally the calibration procedure suggested by [3]. We propose here an other alternative based on the Tikhonov regularization.

The objective of the POD-based model is to represent, as accurately as possible, the dynamics given by the POD temporal eigenfunctions. It is then natural to seek the coefficients $\mathbf{y}$ which minimize the error

$$\mathbf{e}(\mathbf{y}, t) = \dot{\mathbf{a}}^{POD}(t) - \mathbf{f}(\mathbf{y}, \mathbf{a}^{POD}(t)).$$

Since $\mathbf{e} \in \mathbb{R}^{N_{\text{gal}}}$ and is time-dependent, we rather seek to minimize

$$\mathcal{I}(\mathbf{y}) = \langle \|\mathbf{e}(\mathbf{y}, t)\|^2 \rangle_{T_o}, \tag{5}$$

where $\langle \cdot \rangle_{T_o}$ is a time average operator over $[0, T_o]$ ($T_o \leq T$) and $\| \cdot \|$ is a norm of $\mathbb{R}^{N_{\text{gal}}}$. $\langle \cdot \rangle_{T_o}$ corresponds to the arithmetic time-average on N equally spaced elements of the interval $[0, T_o]$:

$$\langle g(t) \rangle_{T_o} = \frac{1}{N} \sum_{k=1}^{N} g(t_k) \quad \text{with} \quad t_k = (k-1)\Delta t \quad \text{and} \quad \Delta t = \frac{T_o}{N-1}.$$

The idea to minimize (5) seems natural because it is equivalent to impose that the temporal POD eigenfunctions $\mathbf{a}^{POD}$ are solutions of the flow model given by $\mathbf{f}$.

Because the error $\mathbf{e}(\mathbf{y}, t)$ is affine, we can then demonstrate that minimizing

$$\mathcal{I}(\mathbf{y}) = \langle \|\mathbf{e}(\mathbf{y}, t)\|^2 \rangle_{T_o}$$

gives rise to the linear system

$$A\mathbf{y} = \mathbf{b}, \tag{6}$$

where $A \in \mathbb{R}^{N_y \times N_y}$ and $\mathbf{b} \in \mathbb{R}^{N_y}$ with $N_y = N_{gal}\left(1 + N_{\text{gal}} + \frac{N_{\text{gal}}(N_{\text{gal}}+1)}{2}\right)$

In practice, the right-hand side is contaminated by approximation errors related to the numerical evaluation of the time-derivatives of the POD eigenfunctions.

The matrix A is ill-conditioned due to the cluster of small singular values of A, i.e., the solution y is potentially very sensitive to perturbations. Hence, its necessary to incorporate further information about the desired solution in order to stabilize the solution. For this reason, we define the regularized solution as the optimal compromise between the residual norm $\mathcal{J}(y)$ and the constraint side $\mathcal{E}(y)$,

$$\min_{\mathbf{y}} \mathcal{J}_\alpha(\mathbf{y}) = \underbrace{\frac{\langle \|\mathbf{e}(\mathbf{y},t)\|_\Lambda^2 \rangle}{\langle \|\mathbf{e}^R(\mathbf{y}^R,t)\|_\Lambda^2 \rangle}}_{\mathcal{J}(\mathbf{y})} + \alpha^2 \underbrace{\frac{\|\mathbf{y}-\mathbf{y}^R\|_\Pi^2}{\|\mathbf{y}^R\|_\Pi^2}}_{\mathcal{E}(\mathbf{y})} \tag{7}$$

where $\Pi \in R^{P\times P}$ is a symmetric matrix. Note that the choice of Π gives a relative importance of each polynomial coefficient and $\mathbf{y}^R$ is the vector of the coefficients obtained directly by Galerkin Projection.

In this formulation, the regularization parameter α controls the weight given to the minimization of $\mathcal{J}(y)$ relative to $\mathcal{E}(y)$. Clearly, a large amount of regularization corresponds to a large value of α and a small amount of regularization corresponds to a small value of α. The subject of α is to find a good compromise between $\mathcal{J}(y)$ and $\mathcal{E}(y)$.

We use here the same analogy as the so-called L-curve method [9] by displaying the compromise between $\mathcal{J}(\mathbf{y})$ and $\mathcal{E}(\mathbf{y})$ in log-log scale. the curve $(\log(\mathcal{J}(\mathbf{y})), \log(\mathcal{E}(\mathbf{y})))$ is expected to have the shape of the letter "L" if the problem is ill-conditioned and contains noise . Its common for ill-conditionned problem that a radical growth of $\mathcal{E}(\mathbf{y})$ occurs when the regularization parameter α gets small. A reasonable solution should lie the vicinity of the "corner", where $\mathcal{E}(\mathbf{y})$ is about to start growing and $\mathcal{J}(\mathbf{y})$ almost remain fix. The corner may be defined as the point where the curve $(\log(\mathcal{J}(\mathbf{y})), \log(\mathcal{E}(\mathbf{y})))$ has its maximum curvature. The figure 3 illustrates this concept and gives the optimal value of α.

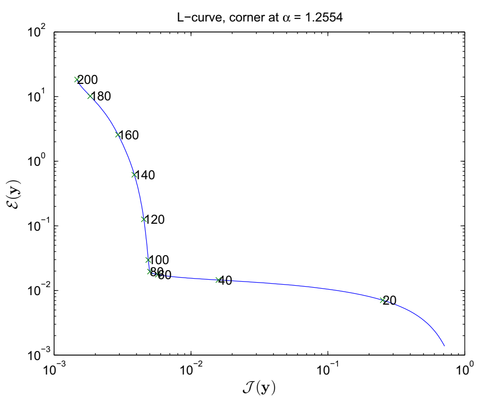
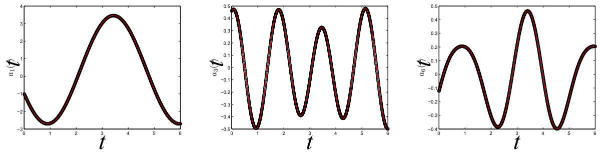


Fig. 3. L-curve corresponding to the minimization of $\mathcal{J}(y)$.

To show the effectivness of the calibrated reduced-order model, we compare in figure 4, the temporal evolutions of the POD modes with those predicted by the calibrated reduced-order model. Contrary to the results presented in figure 1 there is no clear difference in the dynamics. The immediate consequence is that the modal energy distribution associated to the calibrated model now corresponds perfectly to the POD energy (see figure 5). The figure 6 shows the modal error obtained by minimization of $\mathcal{I}$.



(a) Mode 1 (b) Mode 3 (c) Mode 6

Fig. 4. Comparison between the temporal evolutions of the projected modes and the predicted modes obtained by the calibrated system.

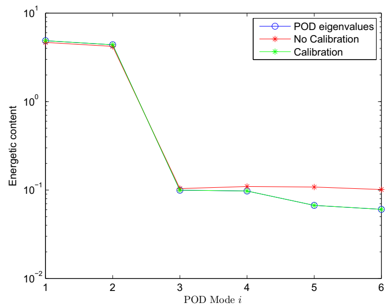


Fig. 5. Comparison between the modal energetic contents obtained before and after calibration.

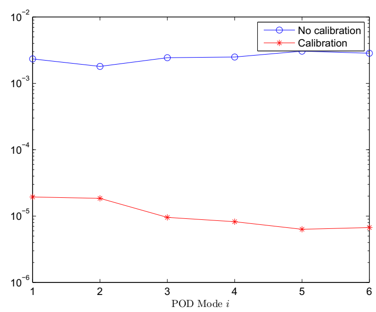


Fig. 6. Modal error obtained by minimization of $\mathcal{I}$.

4 Conclusion

We have presented a numerical method based on regularization technique to improve the accuracy of the POD based reduced order-model. The proposed approach permits to reproduce accurately the dynamic of reference by identifying all the coefficients of the POD ROM. The efficiency of our approach has been demonstrated using a two-dimentional cylindre wake flow. This approach needs to be tested on flow configurations corresponding to more complex dynamics: 3-D turbulent flow obtained by numerical simulations or challenging experimental data.

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