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Vibratory energy exchanges between a system with a chain of Saint-Venant elements and a nonlinear energy sink

C.-H. Lamarque[1] and A. Ture Savadkoohi[1]

ENTPE, Université de Lyon, LGCB and LTDS UMR CNRS 5513, rue Maurice Audin, F-69518 Vaulx-en-Velin, France

Abstract. We present time multi-scale energy exchanges between a system with chain of Saint-Venant elements and a nonlinear energy sink under periodic external forces. A general analytical methodology is presented to detect the invariant manifold of the system at fast time scale and then its equilibrium and fold singularities at slow time scale. The latter development will let us to predict attraction(s) of the dynamical system at slow time scale to be able to design proper nonlinear energy sink devices for the aim of control and/or energy harvesting.

1 Introduction

It is proved that by using the nonlinear properties of some light auxiliary oscillators it is possible to trigger vibratory energy and to control main systems under external excitations. One of these nonlinear system (mainely cubic) is names as nonlinear energy sink (NES) and the phenomenon is called energy pumping [1,2]. Non-smooth nonlinear systems due their efficiency and feasibilities in fabrications and design are of highly interests in industrially [3]. Some research works have been carried to show the effectiveness of non-smooth NES devices in control of main structures [4–8]. However, some researches are carried to analyze and to consider the energy pumping phenomenon between a main non-smooth systems and coupled NES (essentially nonlinear or non-smooth) [9–12]. In the current work we try to consider energy pumping phenomenon between a main structure with chain of Saint-Venant elements in parallel and a NES with a a general nonlinearity. Then we narrow our developments to a main system with two Saint-venant elements in parallel which is coupled by a cubic NES. Organization of the paper is as it follow:

2 The general model of the system

We consider the system which is depicted in Fig. 1: It consists of two oscillators. The first one with mass, stiffness and damping as M, k_0 and $\tilde{\lambda}$, respectively which possesses a set of Saint-Venant elements in parallel with characteristics as $\tilde{k}_j$ and α_j, $j = 1, 2, ..., n$. The second oscillator, namely NES has the mass, stiffness and damping as m ($0 < \epsilon = \dfrac{m}{M} \ll 1$), $\tilde{c}_l$ and $\tilde{\lambda}_1$. The potential of the NES is considered to be nonlinear in general. Governing system of equations of the system reads:

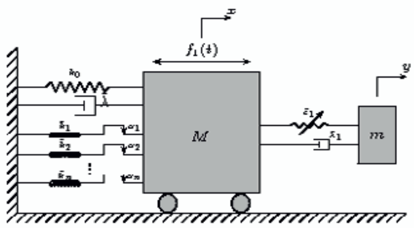

Fig. 1. Two coupled oscillators: the first one with a set of Saint-Venant elements and the second one with nonlinear potential.

$$
\begin{cases}
M\dfrac{d^2x}{dt^2} + \tilde{\lambda}\dfrac{dx}{dt} + \tilde{\lambda}_1(\dfrac{dx}{dt} - \dfrac{dy}{dt}) + k_0 x \\
+ \displaystyle\sum_{j=1}^{n} \tilde{k}_j u_j + \tilde{c}_1 F(x-y) = f_1(t) \\
m\dfrac{d^2y}{dt^2} + \tilde{\lambda}_1(\dfrac{dy}{dt} - \dfrac{dx}{dt}) + \tilde{c}_1 F(y-x) = 0 \\
\left(\dfrac{du_j}{dt} + \beta(\dfrac{u_j}{\eta_j})\right) \ni \dfrac{dx}{dt}, \quad \eta_j = \dfrac{\alpha_j}{\tilde{k}_j}, \quad j = 1, 2, ..., n
\end{cases}
\tag{1}
$$

where F is the nonlinear and odd potential of the NES, i.e. $F(-z) = -F(z)$ (e.g. $F(z) = z^3$) and the β function which is depicted in Fig. 2 can be described as it follows:

$$
\beta(x) = \begin{cases}
\emptyset & \text{if } x \in]-\infty, -1[\cup]1, +\infty[\\
0 & \text{if } x \in]-1, 1[\\
\mathbb{R}_- & \text{if } x = -1 \\
\mathbb{R}_+ & \text{if } x = 1
\end{cases}
\tag{2}
$$

Let us introduce $\tau = t\sqrt{\dfrac{k}{M}} = w_1 t$, $\dfrac{\tilde{\lambda} w_1}{M w_1^2} = \epsilon \lambda_0$, $\dfrac{\tilde{k}_j}{M w_1^2} = \epsilon k_j$, $\dfrac{\tilde{c}_1}{M w_1^2} = \epsilon c_{10}$, $\dfrac{\tilde{\lambda}_1}{M w_1^2} = \epsilon \lambda_{10}$, $\dfrac{f_1(\frac{\tau}{\omega_1})}{M w_1^2} = \epsilon f_{10}\sin(\Omega\tau)$.
We mention that he differential inclusions of the model un-

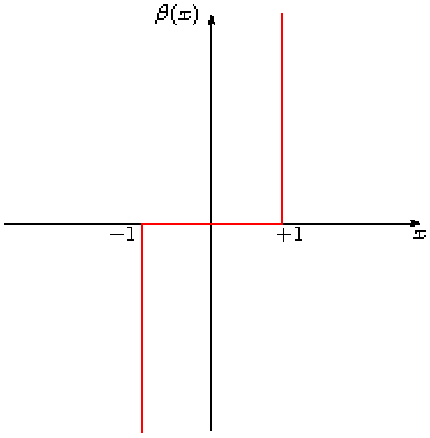

Fig. 2. Graphical representation of the β function in Saint-Venant element.

der consideration come from basic constitutive equations of the Saint-Venant elements as:

$$k_j u_j \in \alpha_j \sigma(\frac{dx}{dt} - \frac{du_j}{dt}), \quad j = 1, 2, ..., n \tag{3}$$

where σ is the graph of the sign:

$$\sigma(z) = \begin{cases} -1 & \text{if } z < 0 \\ [-1, 1] & \text{if } z = 0 \\ 1 & \text{if } z > 0 \end{cases} \tag{4}$$

So one has to take into account that $\frac{d.}{dt} = \omega_1 \frac{d.}{d\tau}$ and

$$\begin{aligned} &\tilde{k}_j u_j \in \alpha_j \sigma\Big(\omega_1(\frac{dx}{d\tau} - \frac{du_j}{d\tau})\Big) \quad j = 1, 2, ..., n \\ \Leftrightarrow\ &\tilde{k}_j u_j \in \alpha_j \sigma\Big((\frac{dx}{d\tau} - \frac{du_j}{d\tau})\Big) \quad j = 1, 2, ..., n \end{aligned} \tag{5}$$

Finally Eqs. (1) are equivalent to

$$\begin{cases} \frac{d^2x}{d\tau^2} + \epsilon\lambda_0 \frac{dx}{d\tau} + \epsilon\lambda_{10}(\frac{dx}{d\tau} - \frac{dy}{d\tau}) + x + \\ \epsilon \sum_{j=1}^{n} k_j u_j + \epsilon c_{10} F(x-y) \\ = \epsilon f_{10} \sin(\Omega\tau) \\ \epsilon \frac{d^2y}{d\tau^2} + \epsilon\lambda_{10}(\frac{dy}{d\tau} - \frac{dx}{d\tau}) + \epsilon c_{10} F(y-x) = 0 \\ \Big(\frac{du_j}{d\tau} + \beta(\frac{u_j}{\eta_j})\Big) \ni \frac{dx}{d\tau}, \quad \eta_j = \frac{\alpha_j}{\tilde{k}_j}, \quad j = 1, 2, ..., n \end{cases} \tag{6}$$

Let us introduce coordinates of the center of mass and relative displacement via

$$\begin{cases} v = x + \epsilon y \\ w = x - y \end{cases} \Leftrightarrow \begin{cases} x = \frac{v + \epsilon w}{1+\epsilon} \\ y = \frac{v - w}{1-\epsilon} \end{cases} \tag{7}$$

System (6) becomes:

$$\begin{cases} \frac{d^2v}{d\tau^2} + \frac{\epsilon\lambda_0}{1+\epsilon}(\frac{dv}{d\tau} + \epsilon\frac{dw}{d\tau}) + \frac{v+\epsilon w}{1+\epsilon} + +\epsilon\sum_{j=1}^{n} k_j u_j \\ = \epsilon f_{10} \sin(\Omega\tau) \\ \frac{d^2w}{d\tau^2} + \frac{\epsilon\lambda_0}{1+\epsilon}(\frac{dv}{d\tau} + \epsilon\frac{dw}{d\tau}) + \frac{v+\epsilon w}{1+\epsilon} + +\epsilon\sum_{j=1}^{n} k_j u_j \\ +(1+\epsilon)(\lambda_{10}\frac{dw}{d\tau} + c_{10}F(w)) = \epsilon f_{10}\sin(\Omega\tau) \\ \Big(\frac{du_j}{d\tau} + \beta(\frac{u_j}{\eta_j})\Big) \ni \frac{1}{1+\epsilon}(\frac{dv}{d\tau} + \epsilon\frac{dw}{d\tau}), \\ \quad \eta_j = \frac{\alpha_j}{\tilde{k}_j}, \quad j = 1, 2, ..., n \end{cases} \tag{8}$$

3 System behavior around $1:1$ resonance

Let us set $T = \Omega\tau$ and $\cdot = \frac{d}{d\tau}$. We introduce following complex variables [13] to the system:

$$\begin{aligned} &\phi_1 e^{\imath T} = \Omega(\dot{v} + \imath v) \quad , \phi_1^* e^{-\imath T} = \Omega(\dot{v} - \imath v) \\ &\phi_2 e^{\imath T} = \Omega(\dot{w} + \imath w) \quad , \phi_2^* e^{-\imath T} = \Omega(\dot{w} - \imath w) \\ &\text{for} \quad j = 1, 2, ..., n \\ &\phi_{2+j} e^{\imath T} = \Omega(\dot{u}_j + \imath u_j) \ , \phi_{2+j}^* e^{-\imath T} = \Omega(\dot{u}_j - \imath u_j) \end{aligned} \tag{9}$$

with $\imath^2 = -1$. To investigate $1:1$ resonance, we consider $\Omega = 1 + \sigma\epsilon$.
We consider only equations obtained by Galerkin method and truncated Fourier series: indeed we take into account only first harmonic $e^{\imath T}$ for each equation. To calculate corresponding Fourier coefficients we assume that ϕ_l and ϕ_l^* $(l = 1, 2, ..., n+j)$ do not depend on T: we will either verify this assumption during the multiple scales analysis, or we will assume that after a transient long enough ϕ_l and ϕ_l^* $(l = 1, 2, ..., n+j)$ reach to an "asymptotic state" independent of T. Nevertheless we also keep $\dot{\phi}_l$ and $\dot{\phi}_2$ in the equations. Then we obtain following system:

$$\begin{aligned} &\Omega\dot{\phi}_1 - \frac{\Omega}{2\imath}\phi_1 + \frac{\epsilon\lambda_0(\phi_1 + \epsilon\phi_2)}{2(1+\epsilon)} + \frac{\phi_1 + \epsilon\phi_2}{2\imath\Omega(1+\epsilon)} + \epsilon\frac{\sum_{j=1}^{n} k_j\phi_{j+2}}{2\Omega\imath} \\ &= \epsilon\frac{f_{10}}{2\imath} \\ &\Omega\dot{\phi}_2 - \frac{\Omega}{2\imath}\phi_2 + \frac{\epsilon\lambda_0(\phi_1 + \epsilon\phi_2)}{2(1+\epsilon)} + \frac{\phi_1 + \epsilon\phi_2}{2\imath\Omega(1+\epsilon)} + \frac{\sum_{j=1}^{n} k_j\phi_{j+2}}{2\Omega\imath} \\ &+(1+\epsilon)(c_{10}\mathcal{F} + \frac{\lambda_{10}}{2}\phi_2) = \epsilon\frac{f_{10}}{2\imath} \\ &\phi_{j+2} = \frac{\phi_1 + \epsilon\phi_2}{(1+\epsilon)\pi}\xi_j\Big(\frac{|\phi_1 + \epsilon\phi_2|}{(1+\epsilon)\Omega}\Big), \quad j = 1, 2, ..., n \end{aligned} \tag{10}$$

where

$$\mathcal{F} = \frac{1}{2\pi}\int_0^{2\pi} e^{-\imath T} F\Big(\frac{\phi_1 e^{\imath T} - \phi_2^* e^{-\imath T}}{2\imath\Omega}\Big)\mathrm{d}T \tag{11}$$

and $\xi_j(z)(\forall z \in \mathbb{R}_+,\ j = 1, 2, ..., n)$ reads:

$$\xi_j(z) = \begin{cases} \pi & \text{if } z \leqslant \eta_j \\ \theta + e^{-\imath\theta}\sin(\theta) - 4e^{-\imath\frac{\theta}{2}}\sin(\frac{\theta}{2}) - \frac{4\eta_j}{z}e^{-\imath(\theta+\frac{\pi}{2})} \\ \quad \text{if } z > \eta_j \end{cases} \tag{12}$$

with

$$\theta = \arccos(1 - \frac{2\eta_j}{z}) \tag{13}$$

Now a multiple scale approach [14] with a small (given) parameter ϵ is presented by considering fast time $T_0 = T$, and slow times $T_l = \epsilon^l T$, $l = 1, 2, ...$ so that

$$\frac{d}{dT} = \frac{d}{dT_0} + \epsilon\frac{d}{dT_1} + \epsilon^2\frac{d}{dT_2} + ... \tag{14}$$

3.1 Fast time scale: ϵ^0-order of the system

At ϵ^0 order, following following equations can be derived from the system of equations (10):

$$\frac{\partial\phi_1}{\partial\tau_0} = 0 \Rightarrow \phi_1 = \phi_1(T_1) \tag{15}$$

$$\frac{\partial\phi_2}{\partial\tau_0} + \frac{\phi_1 - \phi_2}{2\imath} + c_{10}\mathcal{F} + \frac{\lambda_{10}}{2}\phi_2 = 0 \tag{16}$$

$$\phi_{j+2} = \frac{\phi_1}{\pi}\xi_j(|\phi_1|), \quad j = 1, 2, ..., n \tag{17}$$

We can see from equations that ϕ_1 is a constant versus $T_0 = T$, as ϕ_{j+2}, $j = 1, 2, ..., n$ so that the assumption for the calculation of Fourier coefficients of $e^{\imath T} = e^{\imath T_0}$ are verified a posteriori. For ϕ_2, we can not claim the same property. This is why we process as it follows: we assume that when $T_0 \to \infty$, ϕ_2 reaches an asymptotic equilibrium governed by a manifold called T_0 invariant. Then we have:

$$\frac{\phi_1 - \phi_2}{2\imath} + c_{10}\mathcal{F} + \frac{\lambda_{10}}{2}\phi_2 = 0 \tag{18}$$

so that implicitly ϕ_2 may depend on T_1 now, but after long T_0. So we study modulation of the dynamics around periodic solution depending on time T_0 associated to the T_0-invariant. Let us also notice that equations for ϕ_{j+2}, $j = 1, 2, ..., n$ are governed by first order differential equations.

3.2 Slow time scale: ϵ^1-order of the system and modulations around T_0-invariant

The ϵ^1 order of the first equation of system (10) gives:

$$\frac{d\phi_1}{dT_1} + \frac{\lambda_0}{2}\phi_1 + \frac{\phi_2}{2\imath} - \frac{2\sigma + 1}{2\imath}\phi_1 + \frac{\sum_{j=1}^{n} k_j\phi_{j+2}}{2\imath} = \frac{f_{10}}{2\imath} \tag{19}$$

Let us substitute solutions obtained at ϵ^0 order for ϕ_{j+2}, $j = 1, 2, ..., n$ and T_0-invariant. We write Eq. (18) in the general form:

$$\phi_1 = H(\phi_2, \phi_2^*) \tag{20}$$

We introduce polar form for ϕ_j, $j = 1, 2, ..., n + 2$ as it follows:

$$\phi_j = N_j e^{\imath\delta_j},\ N_j \in \mathbb{R}_+,\ \delta_j \in \mathbb{R} \tag{21}$$

From relation (20) it is clear that we can obtain two explicit analytical solutions providing N_1 and δ_1 as functions of N_2 and δ_2:

$$\begin{aligned} N_1 &= H_1(N_2, \delta_2) \\ \delta_1 &= H_2(N_2, \delta_2) \end{aligned} \tag{22}$$

From the Eq. (17) we have:

$$N_{j+2}e^{\imath\delta_{j+2}} = \frac{N_1}{\pi}e^{\imath\delta_1}\xi_j(N_1), \quad j = 1, 2, ..., n \tag{23}$$

or

$$N_{j+2}e^{\imath(\delta_{j+2}-\delta_1)} = \frac{N_1}{\pi}\xi_j(N_1), \quad j = 1, 2, ..., n \tag{24}$$

so that

$$N_{j+2} = \frac{N_1}{\pi}|\xi_j(N_1)|, \quad j = 1, 2, ..., n \tag{25}$$

and δ_{j+2} depends on N_1 and δ_1, let us write

$$\delta_{j+2} = \rho_j(N_1, \delta_1), \quad j = 1, 2, ..., n \tag{26}$$

From (19) we have

$$\begin{aligned} &\frac{\partial N_1}{\partial T_1} + \imath N_1\frac{\partial\delta_1}{\partial T_1} + (\frac{\lambda_0}{2} - \frac{2\sigma + 1}{2\imath})N_1 + \frac{N_2}{2\imath}e^{\imath(\delta_2-\delta_1)} \\ &+\frac{\sum_{j=1}^{n} k_j\frac{N_1}{\pi}\xi_j(N_1)}{2\imath} = \frac{f_{10}}{2\imath}e^{-\imath\delta_1} \end{aligned} \tag{27}$$

Introducing real and imaginary parts of ξ,

$$\xi_j(N_1) = \xi_{jr}(N_1) + \imath\xi_{ji}(N_1), \quad j = 1, 2, ..., n \tag{28}$$

finally one can obtain:

$$\begin{cases} \frac{\partial N_1}{\partial T_1} + \frac{\lambda_0}{2}N_1 + \frac{N_2}{2}\sin(\delta_2 - \delta_1) \\ +\frac{\sum_{j=1}^{n} k_j\frac{N_1}{\pi}\xi_{ji}(N_1)}{2} = -\frac{f_{10}}{2}\sin(\delta_1) \\ N_1\frac{\partial\delta_1}{\partial T_1} + \frac{2\sigma + 1}{2}N_1 - \frac{N_2}{2}\cos(\delta_2 - \delta_1) \\ -\frac{\sum_{j=1}^{n} k_j\frac{N_1}{\pi}\xi_{jr}(N_1)}{2} = -\frac{f_{10}}{2}\cos(\delta_1) \end{cases} \tag{29}$$

Then from (22) we obtain a linear system in $\frac{\partial N_2}{\partial T_1}$ and $\frac{\partial\delta_2}{\partial T_1}$:

$$\begin{cases} \frac{\partial H_1}{\partial N_2}\frac{\partial N_2}{\partial T_1} + \frac{\partial H_1}{\partial\delta_2}\frac{\partial\delta_2}{\partial T_1} - m_1 = 0 \\ H_1(\frac{\partial H_2}{\partial N_2}\frac{\partial N_2}{\partial T_1} + \frac{\partial H_2}{\partial\delta_2}\frac{\partial\delta_2}{\partial T_1}) - m_2 = 0 \end{cases} \tag{30}$$

where

$$-m_1 = \frac{\lambda_0}{2}H_1 + \frac{N_2}{2}\sin(\delta_2 - H_2) + \frac{\sum_{j=1}^{n} k_j \frac{H_1}{\pi}\xi_{ji}(H_1)}{2} + \frac{f_{10}}{2}\sin(\delta_1)$$

$$-m_2 = \frac{2\sigma+1}{2}H_1 - \frac{N_2}{2}\cos(\delta_2 - H_2) - \frac{\sum_{j=1}^{n} k_j \frac{H_1}{\pi}\xi_{jr}(H_1)}{2} + \frac{f_{10}}{2}\cos(\delta_1) \quad (31)$$

Finally solving equations (29), following system is obtained:

$$\frac{\partial N_2}{\partial T_1} = \frac{\tilde{f}_1(N_2,\delta_2)}{\tilde{g}(N_2,\delta_2)}$$
$$N_2\frac{\partial \delta_2}{\partial T_1} = \frac{\tilde{f}_2(N_2,\delta_2)}{\tilde{g}(N_2,\delta_2)} \quad (32)$$

where

$$\tilde{f}_1(N_2,\delta_2) = H_1\frac{\partial H_2}{\partial \delta_2}m_1 - \frac{\partial H_1}{\partial \delta_2}m_2$$

$$\tilde{f}_2(N_2,\delta_2) = N_2\left(\frac{\partial H_1}{\partial N_2}m_2 - H_1\frac{\partial H_2}{\partial N_2}m_1\right) \quad (33)$$

$$\tilde{g}(N_2,\delta_2) = H_1\left(\frac{\partial H_1}{\partial N_2}\frac{\partial H_2}{\partial \delta_2} - \frac{\partial H_2}{\partial N_2}\frac{\partial H_1}{\partial \delta_2}\right)$$

The analysis of the dynamical behavior corresponding to a modulation at 1 : 1 resonance around the T_0-invariant is given by:

- geometry of the T_0-invariant in the N_1, N_2 and δ_2 space associated to the relation $N_1 = H_1(N_2,\delta_2)$.
- fixed points of the reduced system (32) are given by:

$$\begin{cases} f_1(N_2,\delta_2) = 0,\ f_2(N_2,\delta_2) = 0 \\ g_1(N_2,\delta_2) \neq 0,\ g_2(N_2,\delta_2) \neq 0 \end{cases} \quad (34)$$

  if f_1, f_2, g_1 and g_2 correspond to numerators and denominators of (32).
- singular points of the reduced system (32) are given by:

$$\begin{cases} f_1(N_2,\delta_2) = 0,\ f_2(N_2,\delta_2) = 0 \\ g_1(N_2,\delta_2) = 0,\ g_2(N_2,\delta_2) = 0 \end{cases} \quad (35)$$

  if f_1, f_2, g_1 and g_2 correspond to numerators and denominators of (32). Singular points are potentially associated to bifurcations.

4 A numerical example

Let us choose a cubic NES with following characteristics: Let us choose $n = 2$ and

$$F(z) = z^3 \quad (36)$$

in such a case, we have

$$\mathcal{F} = \frac{1}{2\imath}G(|\phi_2|^2)\phi_2 \quad (37)$$

with

$$G(x) = \frac{3c_{10}}{4}x\ ,\quad x \geq 0 \quad (38)$$

and we consider two parallel Saint-Venant elements. It can be proved that is such a case $g_2 \neq 0$. Let us set $c_{10} = 1$, $\lambda_{10} = 0.1$, $\lambda_0 = 0.1$, $\eta_1 = 0.1$, $\eta_2 = 0.15$, $k_1 = 1$, $k_2 = 2$, $\epsilon = 0.001$. We consider external forcing term as $f_{10} = 0.7$. Euler's scheme [3,9] with time steps as $\Delta\tau = 10^{-4}$ is endowed for solving system of equations (6). Assumed initial conditions are $x(0) = 0$ and $y(0) = \dot{x}(0) = \dot{y}(0) = u_1(0) = u_2(0) = 0$. Here present the behavior of the system under higher external forcing terms $f_{10} = 0.7$. Predictions of all possible dynamics of the system until reaching to the infinity of the T_1 time scale are shown in Fig. 3. It is seen that the system has two fold singularities of the first fold line N_{21}, namely points 1 and 2 two equilibrium points 3 and 4 and another equilibrium point between two fold lines of the system (unstable area) namely point no. 5. Invariant manifold of the system with corresponding numerical results are presented in Fig. 4. It is seen that the system presents persistence direct and reverse bifurcations between its stability borders. This behavior will be more visible by looking the histories of system amplitudes which are obtained by numerical integration and are illustrated in Fig. 5. This behavior is named as strongly modulated response (SMR) which is due to the existence of fold singularities on the fold line(s) of the system [7,15]. Phase portraits of the reduced system (Eq. (32)) around fold singular points no. 1 and 2 are presented in Figs. 6 and 7 shows that these singular points are in the form of saddle and nodes on the fold line of the system (N_{21}). During SMR both oscillators and all of their components present beating response: displacement histories of two oscillators which are depicted in Fig. 8 and also histories of internal variables of Saint-Venant elements which are presented in Fig. 9 show not only beating responses of all components of two oscillators during SMR but also well activations of Saint-Venant elements during energy exchanges between them. The SMR of an optimized designed system is a very desirable behavior from passive control and also energy harvesting view points since both oscillators continue to exchange the energy with large intervals of energy change for the NES and very small energy intervals for the the main system. The system possesses two equilibrium points namely points no. 3 and 4 (see Fig. 3). It can be attracted by one of these points after a very long time at T_1 time scale or during higher time scales (T_2, T_3, ...). Due to costly simulation time we did not run it for very long time scales.

5 Conclusions

Vibratory energy exchange between two nonlinear systems is studied: The first system contains a chain of parallel Saint-Venant elements. The second oscillator is a nonlinear energy sink with a very small mass with respect to the first oscillator and with a general nonlinear potential. The analysis of the system at fast and also time scales let us obtain its invariant manifold and predict its equilibrium and singular points. These predictions lead us to explain different responses of the system which include bifurcation, attractions to periodic regime and/or strongly modulated responses. The developed technique can be endowed for

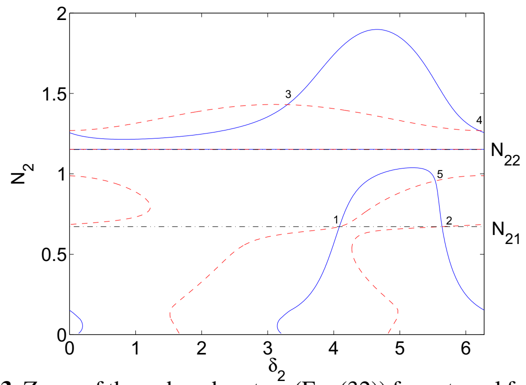

Fig. 3. Zeros of the reduced system (Eq. (32)) for external forcing term $f_0 = 0.7$: $f_1 = 0$ (—), $f_2 = 0$ (– – –), $g_1 = 0$(–· –· –, i.e. fold lines N_{21} and N_{22}).

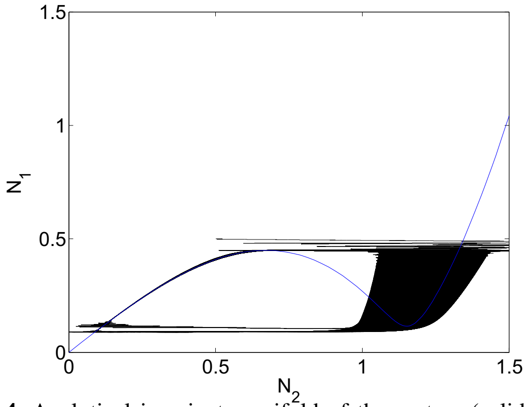

Fig. 4. Analytical invariant manifold of the system (solid blue line) and corresponding numerical results for the system with $f_0 = 0.7$.

designing nonlinear energy sink devices for passively controlling main system which contain friction terms.

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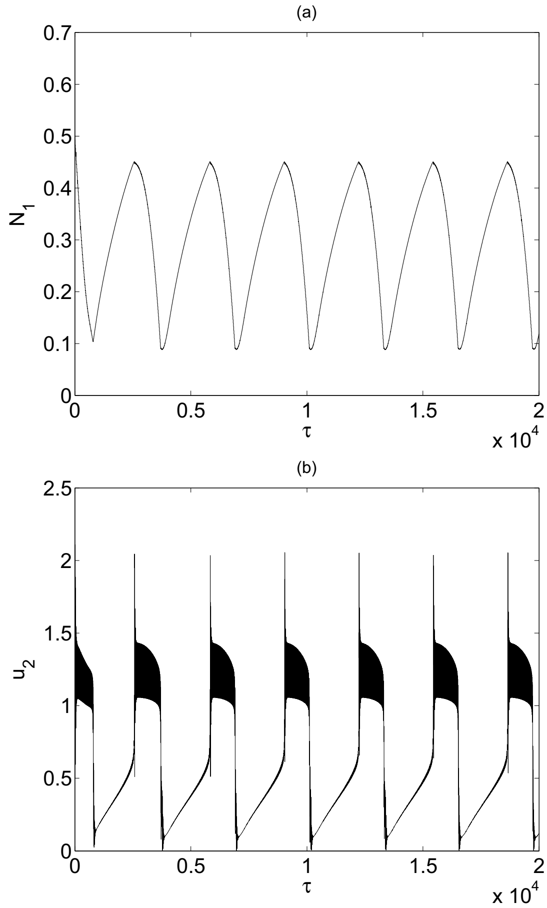

Fig. 5. Numerical values of system amplitudes for the system with $f_0 = 0.7$: a) N_1; b) N_2.

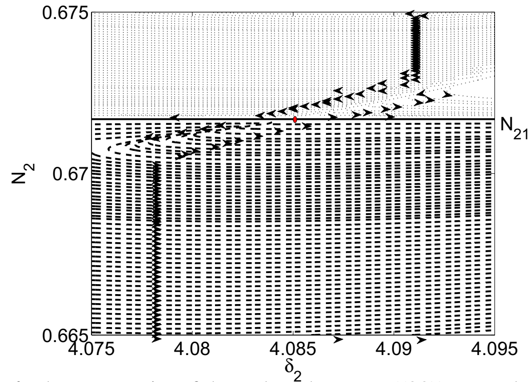

Fig. 6. Phase portraits of the reduced system ((32)) around singular point no. 1 (saddle) with external forcing term $f_0 = 0.7$.

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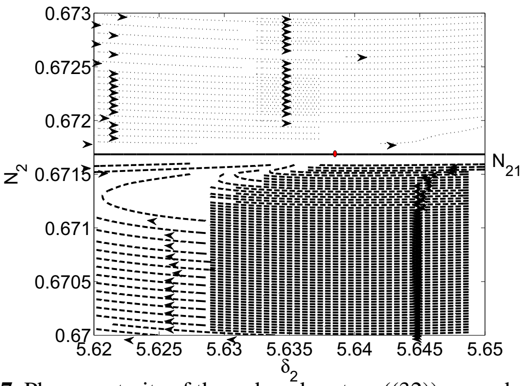

Fig. 7. Phase portraits of the reduced system ((32)) around singular point no. 2 (node) with external forcing term $f_0 = 0.7$.

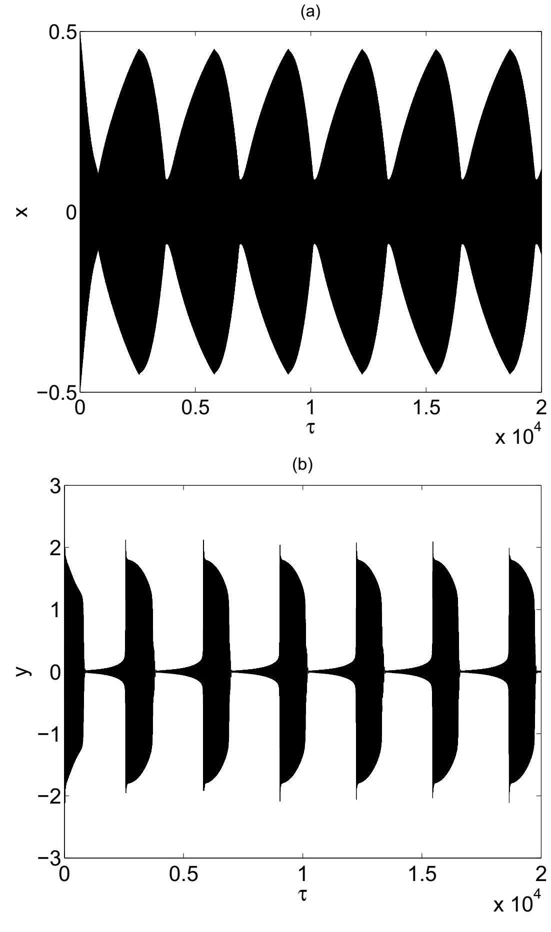

Fig. 8. Numerical values of system displacements for the system with $f_0 = 0.7$: a) x; b) y.

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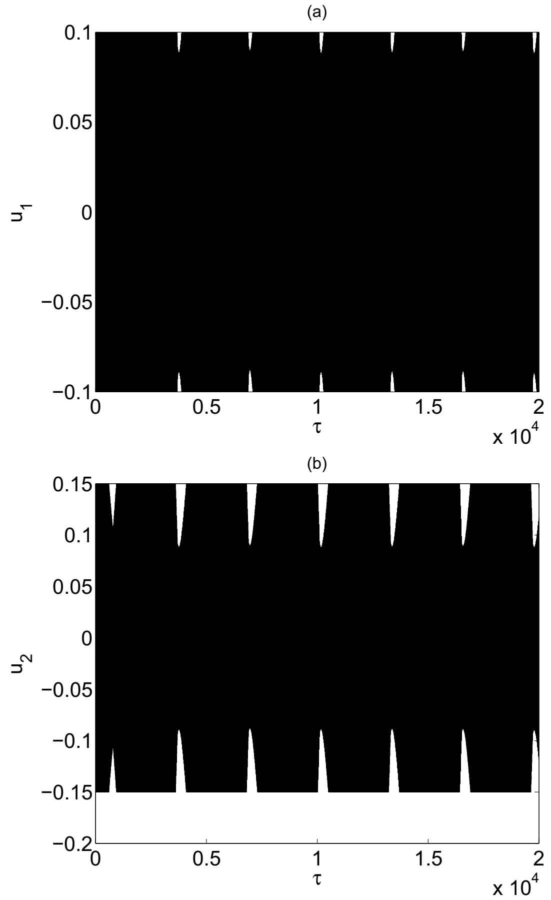

Fig. 9. Numerical values of internal variables of the Saint-Venant element for the system with $f_0 = 0.7$: a) u_1; b) u_2.

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