MATEC Web of Conferences **1**, 10012 (2012)
DOI: 10.1051/matecconf/20120110012

Synthesis of multi-input Volterra systems by a topological assemblage scheme

L. Carassale[1], A. Kareem[2]

[1]Dept. of Civil, Environmental and Architectural Engineering, University of Genova, Italy
[2]NatHaz Modelling Laboratory, University of Notre Dame, IN, USA

Abstract. The Volterra series expansion is widely employed to represent the input-output relationship of nonlinear dynamical systems. Such a representation is based on the Volterra frequency-response functions (VFRF), which can be calculated from the equation governing the system by the so-called harmonic probing method. This operation is straightforward for simple systems, may reach a prohibitive level of complexity for multiple-input systems when the calculation of a high-order VFRF is required. An alternative technique for the evaluation of the VFRFs of multiple-input systems is here presented generalizing an existing technique originally limited to the scalar case. A 2-dof mechanical example is used to illustrate the application of the technique.

1 Introduction

The Volterra series is a mathematical tool widely employed for the representation of the input-output relationship of nonlinear dynamical systems. It is based on the expansion of the nonlinear operator representing the system into a series of homogeneous operators, formally similar to the Duhamel integral usually employed for the analysis of linear systems. Such integrals are multi-dimensional ones and are completely defined given the Volterra kernels, i.e. the multi-dimensional generalization of the impulse-response function [1]. An alternative representation is provided by the Volterra frequency-response functions (VFRF), which represents the frequency-domain counterparts of the Volterra kernels and can be reviewed as a generalization of the usual frequency-response function. A fairly large class of dynamical systems can be treated according to these concepts, and therefore represented in terms of VFRFs.

When an analytical model of the dynamical system is available (e.g. through a non-linear differential equation), the VFRFs are traditionally calculated by means of the harmonic probing method, consisting in evaluating analytically the response of the system excited by products of harmonic functions with different frequencies [2]. This approach to synthesize Volterra systems is straightforward when the governing differential equation is reasonably simple, but may reach a prohibitive level of computational complexity when dealing with high-order nonlinear systems or for the calculation of a high-order VFRF [3]. The practical application of the harmonic probing technique becomes even more complicated when the considered system has multiple input and the possible combinations of input harmonics to be probed increase dramatically [4].

An approach alternative to the harmonic probing has been presented in [5] for the case of scalar systems. It involves the representation of a complex dynamical system by an assemblage of simple operators for which VFRFs are readily available (essentially polynomial operators and derivatives). The topology of the assemblage is determined by the mathematical structure of the governing equation and the VFRFs of the composite system are evaluated by composing the VFRFs of the elementary building blocks by means of algebraic rules.

The present paper discusses the extension of the aforementioned approach to the synthesis of multiple-input/multiple-output systems. The adoption of a vectorial format and the use of the Kronecker algebra enable the definition of assemblage rules formally similar to the ones adopted for scalar systems.

The use of the proposed procedure is illustrated by synthesizing a simple non-linear system, calculating its VFRFs of any order. The accuracy of the Volterra series representation is discussed through a numerical application involving the computation of both deterministic and stochastic response.

2 Theoretical background

Let us consider the nonlinear system represented by the following equation:

$$\mathbf{x}(t)=\mathscr{H}\left[\mathbf{u}(t)\right] \tag{1}$$

$\mathbf{u}(t)$ and $\mathbf{x}(t)$ being vectors with size n and m, respectively, representing the input and the output; t is the time. If the operator $\mathscr{H}[\bullet]$ is time-invariant and has finite-memory, its output $\mathbf{x}(t)$ can be expressed, far enough from the initial conditions, through the Volterra series expansion [1]:

$$\mathbf{x}(t)=\sum_{j=0}^{\infty}\int_{\boldsymbol{\tau}_j\in\mathbb{R}^j}\mathbf{h}_j(\boldsymbol{\tau}_j)\prod_{r=1}^{j}\mathbf{u}(t-\tau_r)\,\mathrm{d}\boldsymbol{\tau}_j \tag{2}$$

where $\boldsymbol{\tau}_j=[\tau_1\ldots\tau_j]^{\mathrm{T}}$ is a vector containing the j integration variables; the functions $\mathbf{h}_j$ have values in $\mathbb{R}^{m\times n^j}$ and are called Volterra kernels. The product operator is interpreted as a sequence of Kronecker products, i.e.:

$$\prod_{r=1}^{j}\mathbf{u}(t-\tau_r)=\mathbf{u}(t-\tau_1)\otimes\cdots\otimes\mathbf{u}(t-\tau_j) \tag{3}$$

The 0^{th}-order term of the Volterra series, $\mathbf{h}_0$, is a constant independent of the input; the 1^{st}-order term is the convolution integral typical of the linear dynamical systems, with $\mathbf{h}_1$ being the impulse response function. The higher-order terms are multiple convolutions involving products of the input values for different delay times.

A Volterra system is entirely determined by its constant output and its Volterra kernels. An alternative representation is provided, in the frequency domain, by the Volterra frequency-response functions (VFRF), the multi-dimensional Fourier transforms of the Volterra kernels:

$$\mathbf{H}_j(\boldsymbol{\Omega}_j)=\int_{\boldsymbol{\tau}_j\in\mathbb{R}^j}\mathrm{e}^{-\mathrm{i}\boldsymbol{\Omega}_j^{\mathrm{T}}\boldsymbol{\tau}_j}\mathbf{h}_j(\boldsymbol{\tau}_j)\,\mathrm{d}\boldsymbol{\tau}_j \tag{4}$$

where $\boldsymbol{\Omega}_j=[\omega_1\ldots\omega_j]^{\mathrm{T}}$ is a vector containing the j circular frequency values corresponding to $\tau_1,\ldots,\tau_j$ in the Fourier transform pair. The VFRFs are functions with values in $\mathbb{C}^{m\times n^j}$.

3 VFRFs for composite systems

Rules for the parallel and series assemblage of Volterra systems, as well as for their product and power are briefly discussed. The proofs are not reported for reason of space, but can be easily derived following the procedures applied in [5].

3.1 Parallel assemblage of Volterra systems

When a nonlinear operator $\mathscr{H}$ is realized by the parallel assemblage (or sum) of the Volterra operators $\mathscr{A}$ and $\mathscr{B}$, i.e. $\mathbf{x}=\mathscr{H}[\mathbf{u}]=\mathscr{A}[\mathbf{u}]+\mathscr{B}[\mathbf{u}]$, then the VFRFs of $\mathscr{H}$ can be obtained as:

$$\mathbf{H}_j(\boldsymbol{\Omega}_j)=\mathbf{A}_j(\boldsymbol{\Omega}_j)+\mathbf{B}_j(\boldsymbol{\Omega}_j)\qquad (j=0,1,\ldots) \tag{5}$$

where $\mathbf{A}_j$ and $\mathbf{B}_j$ are VFRFs of $\mathscr{A}$ and $\mathscr{B}$, respectively.

3.2 Product of Volterra systems

Let us consider a nonlinear operator $\mathscr{H}$ consisting of the Kronecker product of the Volterra systems $\mathscr{A}$ and $\mathscr{B}$, i.e., $\mathbf{x}=\mathscr{H}[\mathbf{u}]=\mathscr{A}[\mathbf{u}]\otimes\mathscr{B}[\mathbf{u}]$. The VFRFs of $\mathscr{H}$ can be obtained as:

$$\mathbf{H}_j(\boldsymbol{\Omega}_j)=\sum_{\alpha_p^{(j,2)}}\mathbf{A}_{\alpha_1^{(j,2)}}\left(\boldsymbol{\theta}_1^{(j,2)}\right)\otimes\mathbf{B}_{\alpha_2^{(j,2)}}\left(\boldsymbol{\theta}_2^{(j,2)}\right) \tag{6}$$

where $\alpha_p^{(j,k)}$ $(p=1,\ldots,k)$ are sequences of numbers such that

$$\alpha_p^{(j,k)}\geq 0 \qquad (p=1,\ldots,k)$$
$$\sum_{p=1}^{k}\alpha_p^{(j,k)}=j \tag{7}$$

and $\boldsymbol{\theta}_1^{(j,k)},\ldots,\boldsymbol{\theta}_k^{(j,k)}$ are vectors with size, respectively, $\alpha_1^{(j,k)},\ldots,\alpha_k^{(j,k)}$ partitioning $\boldsymbol{\Omega}_j$.

These results can be obviously generalized to the iterated Kronecker product, i.e. $\mathbf{x}=\mathscr{H}[\mathbf{u}]=\mathscr{A}[\mathbf{u}]^{[k]}=\mathscr{A}[\mathbf{u}]\otimes\ldots\otimes\mathscr{A}[\mathbf{u}]$ (k times).

$$\mathbf{H}_j(\boldsymbol{\Omega}_j)=\sum_{\alpha_p^{(j,k)}}\prod_{r=1}^{k}\mathbf{A}_{\alpha_r^{(j,k)}}\left(\boldsymbol{\theta}_r^{(j,k)}\right) \tag{8}$$

where the product operator is interpreted as a sequences of Kronecker products.

3.3 Series assemblage of Volterra systems

Let us consider an operator $\mathscr{H}$ consisting of the series combination between the two Volterra systems $\mathscr{A}$ and $\mathscr{B}$, in such a way that the output $\mathbf{y}=\mathscr{A}[\mathbf{u}]$ of the operator $\mathscr{A}$ is the input for the operator $\mathscr{B}$, i.e. $\mathscr{H}=\mathscr{B}\left[\mathscr{A}[\mathbf{u}]\right]$. The VFRFs of $\mathscr{H}$ can be obtained as:

$$\mathbf{H}_j(\boldsymbol{\Omega}_j)=\sum_{k=1}^{N_{\mathscr{B}}}\sum_{\alpha_p^{(j,k)}}\mathbf{B}_k\left(\mathbf{S}^{(j,k)}\boldsymbol{\Omega}_j\right)\prod_{r=1}^{k}\mathbf{A}_{\alpha_r^{(j,k)}}\left(\boldsymbol{\theta}_r^{(j,k)}\right)\ (j\geq 1) \tag{9}$$

where $N_{\mathscr{B}}$ is the order of the system $\mathscr{B}$ and $\mathbf{S}^{(j,k)}$ is a matrix of dimension $k\times j$ defined as:

$$\mathbf{S}^{(j,k)}=\begin{bmatrix}\overbrace{1\cdots1}^{\alpha_1^{(j,k)}} & \overbrace{}^{\alpha_2^{(j,k)}} & & 0\\ & 1\cdots1 & & \\ & & \cdots & \overbrace{}^{\alpha_k^{(j,k)}}\\ 0 & & & 1\cdots1\end{bmatrix} \tag{10}$$

It is worth noting that the j-order term of Eq. (9) contains all the VFRFs $\mathbf{A}_k$ for $k\leq j$ and that, if $\mathbf{A}_0=0$, then all the sequences containing an $\alpha_r^{(j,k)}=0$ do not give any contribution to the sum, thus the VFRF $\mathbf{H}_j$ contains only the VFRFs $\mathbf{B}_k$ with $k\leq j$, while $\mathbf{A}_j$ appears only once, multiplied by $\mathbf{B}_1$. In this case, Eq. (9) can be rewritten in the simplified form:

$$\mathbf{H}_j(\boldsymbol{\Omega}_j) = \mathbf{B}_1(\Sigma\boldsymbol{\Omega}_j)\mathbf{A}_j(\boldsymbol{\Omega}_j) + \\ + \sum_{k=2}^{j}\sum_{\alpha_p^{(j,k)}} \mathbf{B}_k\left(\mathbf{S}^{(j,k)}\boldsymbol{\Omega}_j\right)\prod_{r=1}^{k}\mathbf{A}_{\alpha_r^{(j,k)}}\left(\boldsymbol{\theta}_{\alpha_r^{(j,k)}}\right) \tag{11}$$

where the only term in $\mathbf{A}_j$ has been extracted from the summation and $\Sigma\boldsymbol{\Omega}_j$ is the sum of the components of the vector $\boldsymbol{\Omega}_j$.

If $\mathscr{B}$ is a linear homogeneous operator (i.e., no DC output), then Eq. (9) becomes very simple, resulting

$$\mathbf{H}_j(\boldsymbol{\Omega}_j) = \mathbf{B}_1(\Sigma\boldsymbol{\Omega}_j)\mathbf{A}_j(\boldsymbol{\Omega}_j) \tag{12}$$

while if, on the contrary, $\mathscr{A}$ is a linear homogeneous operator, then Eq. (9) becomes

$$\mathbf{H}_j(\boldsymbol{\Omega}_j) = \mathbf{B}_j(\boldsymbol{\Omega}_j)\prod_{r=1}^{j}\mathbf{A}_1(\omega_r) \tag{13}$$

4 Numerical example

As an example of application of the synthesis techniques discussed above, the 2-dof mechanical system described in [4] (figure 1) is considered. Adopting a vectorial notation, such a system can be represented by the differential equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_1\dot{\mathbf{x}} + \mathbf{C}_2\dot{\mathbf{x}}^{[2]} + \mathbf{C}_3\dot{\mathbf{x}}^{[3]} + \mathbf{K}_1\mathbf{x} + \mathbf{K}_2\mathbf{x}^{[2]} + \mathbf{K}_3\mathbf{x}^{[3]} = \mathbf{u} \tag{14}$$

where the vector $\mathbf{x} = [x_1\ x_2]^T$ contains the displacement of the two masses and $\mathbf{u} = [u_1\ u_2]^T$ the corresponding external forces; $\mathbf{M}$ is the mass matrix, $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ are linear and non-linear damping matrices, $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ linear and non-linear stiffness matrices. The superscript $\bullet^{[k]}$ represents the Kronecker power. With reference to figure 1, the matrices the defining the system are given as:

$$\mathbf{M} = diag(m_1, m_2); \qquad \mathbf{K}_1 = \begin{bmatrix} k_{11}+k_{12} & k_{12} \\ k_{12} & k_{22}+k_{12} \end{bmatrix};$$
$$\mathbf{C}_1 = \begin{bmatrix} c_{11}+c_{12} & c_{12} \\ c_{12} & c_{22}+c_{12} \end{bmatrix}; \quad \mathbf{C}_2 = c_2\begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix}; \tag{15}$$
$$\mathbf{C}_3 = c_3\begin{bmatrix} 1 & -3 & 0 & 3 & 0 & 0 & 0 & 1 \\ -1 & 3 & 0 & -3 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$\mathbf{K}_2$ and $\mathbf{K}_3$ are sparse matrices having the only non-zero entries $[\mathbf{K}_2]_{1,1} = k_2$ and $[\mathbf{K}_3]_{1,1}=k_3$. The numerical values adopted in the calculation are the same as in [4].

The VFRFs of the whole mechanical system can be obtained by using the assemblage rules described in Section 3. To this purpose, the system defined by Eq. (14) must be re-casted considering its non-linear terms as a feedback for the linear part represented by the operator $\mathscr{D}^{-1}$ (figure 2).

Reading the scheme of figure 2 from right to left, the following operatorial equation can be obtained equating the (reversed) linear part of the system, $\mathscr{D}[\mathbf{x}] = \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_1\dot{\mathbf{x}} + \mathbf{K}_1\mathbf{x}$, and the feedback branch:

$$\mathscr{D}[\mathbf{x}] = \mathbf{u} - \mathscr{A}[\mathbf{x}] - \mathscr{B}[\mathbf{x}] \tag{16}$$

where the operators $\mathscr{A}$ and $\mathscr{B}$ are schematically defined in figure 2. Since all these operators have a polynomial structure, their VFRFs can be easily obtained as:

$$\begin{aligned} &\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C}_1 + \mathbf{K}_1 \\ &\mathbf{A}_j(\boldsymbol{\Omega}_j) = \mathbf{K}_j \qquad (j = 2,3) \\ &\mathbf{B}_j(\boldsymbol{\Omega}_j) = i\Sigma\boldsymbol{\Omega}_j\mathbf{C}_j \qquad (j = 2,3) \end{aligned} \tag{17}$$

with $\mathbf{A}_j = \mathbf{B}_j = \mathbf{0}$ for $j \neq 2, 3$.

Assuming that $\mathbf{x}$ is the output of an unknown operator $\mathscr{H}$ (i.e., $\mathbf{x} = \mathscr{H}[\mathbf{u}]$), then Eq. (16) can be rewritten as

$$\mathscr{D}\big[\mathscr{H}[\mathbf{u}]\big] = \mathbf{u} - \mathscr{A}\big[\mathscr{H}[\mathbf{u}]\big] - \mathscr{B}\big[\mathscr{H}[\mathbf{u}]\big] \tag{18}$$

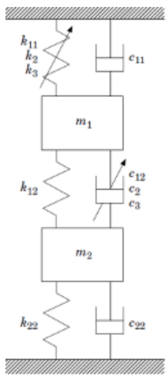

Fig 1. Mechanical system considered in the example (from [4]).

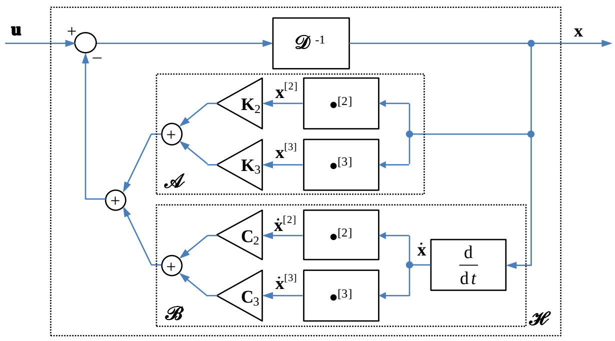

Fig 2. Scheme of the system to be synthesized.

The VFRFs of the left-hand side and of the right-hand side of Eq. (18) can be obtained through the assemblage rules defined in Section 3 and equated order by order. At the 0[th] order, it results $\mathbf{H}_0 = \mathbf{0}$ as the only solution of the non-linear algebraic equation:

$$\mathbf{D}(0)\mathbf{H}_0 = \mathbf{A}_2\mathbf{H}_0^{[2]} + \mathbf{A}_3\mathbf{H}_0^{[3]} \tag{19}$$

At the 1[st] order the VFRF is simply obtained by the inversion of $\mathscr{D}$

$$\mathbf{H}_1(\omega) = \mathbf{D}(\omega)^{-1} \tag{20}$$

and for any order $j \geq 2$ the VFRFs are given as:

$$\mathbf{H}_j(\boldsymbol{\Omega}_j) = \mathbf{D}(\Sigma\boldsymbol{\Omega}_j)^{-1} \sum_{k=2}^{3} \sum_{\alpha_p^{(j,k)}} \left[\mathbf{A}_k + \mathbf{B}_k\left(\mathbf{S}^{(j,k)}\boldsymbol{\Omega}_j\right)\right] \prod_{r=1}^{k} \mathbf{H}_{\alpha_r^{(j,k)}}\left(\boldsymbol{\theta}_r^{(j,k)}\right) \tag{21}$$

in which the j^{th}-order VFRF is obtained as a function of the VFRFs with orders up to $j-1$.
Figure 3 shows the absolute value of the 2^{nd}-order VFRF calculated by Eq. (21).

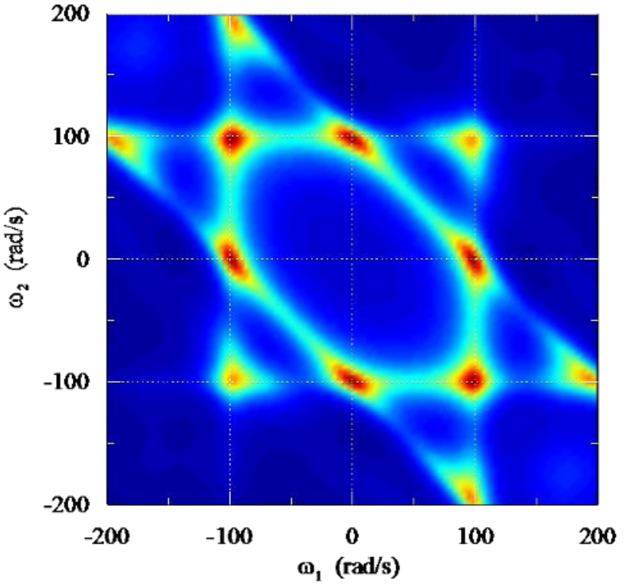

Fig. 3. Absolute value of the 2^{nd}-order VFRF.

Figure 4 shows a 1 s long time history of the response $x_1(t)$ obtained by time-domain integration of the non-linear differential equation and by the frequency-domain calculation based on the Volterra series of order 1, 2 and 3. The input $\mathbf{u}$ is constituted by two uncorrelated, zero-mean, stationary Gaussian random process having constant Spectral Density function (PSD) in the harmonic band between $\omega_1 = 11\pi$ and $\omega_2 = 21\pi$, zero elsewhere and standard deviation $\sigma_{u_1} = \sigma_{u_2} = 1.58$ N. Samples of the input process are generated by a Monte Carlo Simulation (MCS) method.

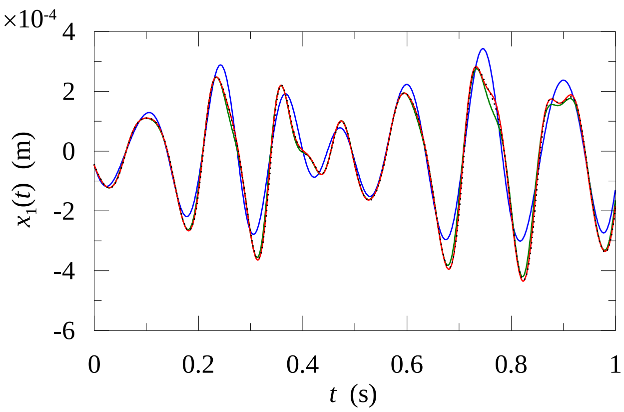

Fig 4. Time history of the displacement component $x_1(t)$. Time domain integration (dots), 1^{st}-order Volterra system (blue), 2^{nd}-order Volterra system (green), 3^{rd}-order Volterra system (red).

Figure 5 shows the PSD of the displacement of x_1. The PSD of the original non-linear model is obtained by time-domain MCS, while the PSDs of the Volterra models are calculated in the frequency domain from the VFRFs [5]. It can be observed that the linear model represents the results of the original system only in the harmonic range in which the excitation is present; the 2^{nd}-order model represents the response up to the frequency $2\omega_2$; the 3^{rd}-order model up to $3\omega_2$.

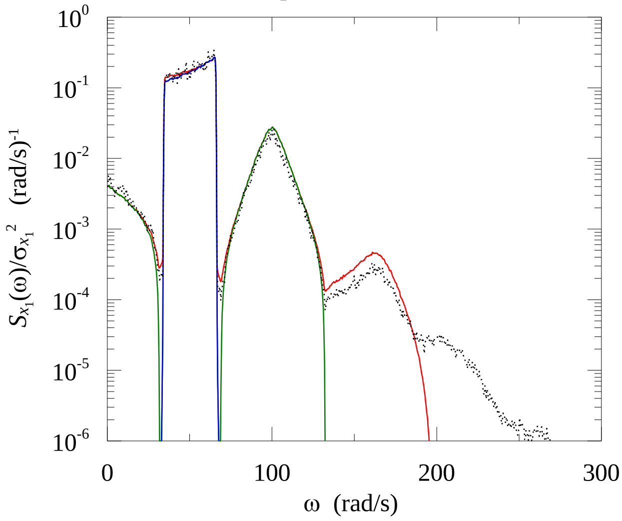

Fig 5. PSD of the displacement component $x_1(t)$. MCS (dots), 1^{st}-order Volterra system (blue), 2^{nd}-order Volterra system (green), 3^{rd}-order Volterra system (red).

5 Conclusions

The idealization of a given dynamical system as an assemblage of elementary operators (building blocks) enables the evaluation of its VFRFs of any order by means of algebraic rules. The rules governing the assemblage of multi-input/multi-output systems are formally analogous to the ones derived for scalar system with the only complication involved in the use of the vectorial notation and the Kronecker algebra. A numerical example demonstrated the application of the proposed technique on a 2-dof mechanical system excited by both deterministic and stochastic loads.

References

1. M. Schetzen, *The Volterra and Weiner theories of nonli-near systems* (John Wiley & Sons, NY, 1980)
2. E. Bedrosian, S.O. Rice, The output of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputs, Proc. IEEE **59**, 12 (1971)
3. J.C. Peyton Jones, Simplified computation of the Volterra frequency response functions of non-linear systems, Mech. Sys. & Sig. Proc., **23** (2007)
4. K. Worden, G. Manson, G.R. Tomlinson, A harmonic probing algorithm for the multi-input Volterra series. J. Sound & Vibr., **201**, 1 (1997)
5. L. Carassale, A. Kareem, Modeling nonlinear systems by Volterra Series, J. Engrg. Mech. ASCE, **136**, 6 (2010)