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Effect of Horizontal Vibration on the Interfacial Instability in a Horizontal Hele-Shaw Cell

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Abstract. The effect of periodic oscillations on the interfacial instability of two immiscible fluids, confined in a horizontal Hele-Shaw cell, is investigated. A linear stability analysis of the basic state leads to a periodic Mathieu oscillator corresponding to the amplitude of the interface. Then, the threshold of parametric instability of the interface is characterized by harmonic or subharmonic periodic solutions. We show that the relevant parameters that control the interface are the Bond number, density ratio, Weber number and amplitude and frequency of oscillations.

1 Introduction

Several works have been carried out to study the interfacial instability of superposed layers subject to periodic oscillations both experimentally and theoretically [1–9]. Vertical oscillations, for instance, can lead both to Faraday instability [1,2], and to suppression of Rayleigh-Taylor instability [3,4]. However, the experimental study carried out by wolf [3] has shown that the horizontal oscillations have also been found to suppress instabilities. Jiang et *al*. [5] have considered two superposed fluid layers, with a free surface, flowing down an inclined plane which oscillates sinusoidally and parallely to the flow direction. They have shown that the onset of instability, in this unstable film flow, can be suppressed in certain parameter regions by imparting oscillations of appropriate amplitudes and frequencies to the plate. In this work, it turns out that the manifestation of instability is enriched by variable stratification of density, viscosity, and the thickness ratio of the two layers. The Kelvin-Helmholtz instability with oscillating flow has been studied first by Kelly [6]. He has focused his study on the stability of an interface between two superposed fluid layers in which the velocity fields are periodic. He has reduced the linear stability problem to a periodic Mathieu equation and has discovered the cases where the oscillations stabilized the unstable shear flow. An application of Kelly's problem [6] has been given by Lyubimov and Cherepanov [7]. They have studied the interfacial instability of two superposed layers of inviscid fluids contained in a rectangular vessel which is subject to horizontal oscillations. The authors have analyzed the behavior of the interface of fluid layers with comparable densities and in a higher frequency vibrational field. They have decoupled the fast oscillatory motion from the mean flow in the limit of high frequencies. In this approach, two parameters have been assumed to be asymptotically small simultaneously, the dimensionless thickness of the viscous skin-layers and the dimensionless amplitude of the vibration. In this limiting case, only the basic instability mode associated with the development of the wave relief, Kelvin-Helmholtz instability, on the interface remains and the possibility of a parametric resonance description disappears. Later, another study similar to that in [6,7], has been carried out by Khenner et *al*. [8] who have treated the linear stability of the interface between two superposed fluids under horizontal oscillations. Similarly to Kelly [6], they have found parametric resonant regions of instability associated with the intensification of the capillary waves at the interface. Recently, the linear stability analysis model carried out by Khenner et *al*.[8] has been extended both theoretically [9] and experimentally [10] by Talib and Juel to include the effect of the viscosities of the fluid layers of finite depth. Talib et *al*. [10] have solved numerically the linear stability problem for an exhaustive range of vibrational to viscous forces ratios and viscosity contrasts. They have shown that the viscous model allows to predict the onset of each mode of instability particularly in the limit of high viscosity contrast. Talib et *al*. [10] have characterized the evolution of the neutral curves from the multiple modes of the parametric-resonant instability to the single frozen wave mode encountered in the limit of practical flows. They have found that the interface has been linearly unstable to a Kelvin-Helmholtz mode and successive parametric-resonance modes. These two modes exhibit opposite dependencies on the viscosity contrast which have been understood by examining the eigenmodes near the interface.

In the present investigation, we consider two immiscible, incompressible viscous liquid layers confined in a horizontal Hele-Shaw cell under horizontal sinusoidal vibration. We thus perform an inviscid linear stability analysis to describe the perturbation of the interface and we focus attention on the effect of periodic oscillations on the threshold of the instability of the interface between two fluids.

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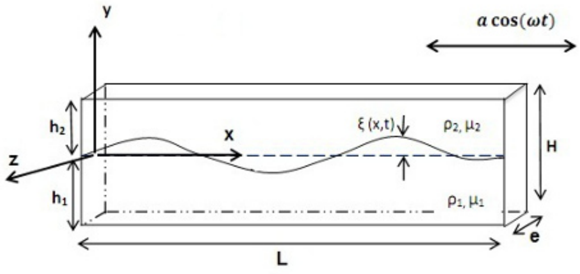

Fig. 1. Scheme arrangement of Hele-Shaw cell

2 Formulation

2.1 Governing equations

Consider two immiscible, incompressible Newtonian liquid layers confined in a horizontal Hele-Shaw cell. The heavy fluid occupies the bottom region of hight h_1 and the light one occupies the upper region of hight h_2. We denote by $H = h_1 + h_2$ the height of the cell, e the distance between the vertical walls and $\epsilon = \frac{e}{H} \ll 1$ the aspect ratio of the cell. The values $z = \pm\frac{e}{2}$ and $y = 0, H$ correspond to the boundaries of the cell, see figure 1. Each fluid layer is characterized by a density ρ_j, a kinematic viscosity, ν_j, where the subscripts $j = 1, 2$ denotes the lower and the upper layer respectively. Suppose that the Hele-Shaw cell is submitted to horizontal oscillatory motion according to the law of displacement $a\cos(\omega t)$ **x**, where a and ω designate, respectively, the displacement amplitude and the dimensional frequency of the oscillatory motion. Therefore, the fluid layers are submitted to two volumic forces: the oscillatory force $-\rho_j a\omega^2 \cos(\omega t)\,\mathbf{x}$ and the gravitational one $\rho_j\,\mathbf{g}$. The denser fluid is placed in the lower layer, so that the configuration is gravitationally stable. under these assumptions, the physical problem is governed by the following set of equations:

$$\nabla.\mathbf{V}_j = 0 \qquad (j = 1, 2) \tag{1}$$

$$\rho_j \frac{d\mathbf{V}_j}{dt} = -\nabla p_j + \mu_j \Delta \mathbf{V}_j + \rho_j a\omega^2 \cos(\omega t)\mathbf{x} + \rho_j \mathbf{g} \tag{2}$$

where p_j is the hydrodynamic pressure and $\mathbf{V}_j = (u_j, v_j)$ the velocity in each fluid layer.

2.2 Base flow solutions

In the basic state, the fluid layers are separated by an interface which is initially planar, horizontal and coincident with the $y = 0$ plan by choice of the coordinate system. However, to determine the basic flow we look for one-component velocity field $\mathbf{V}^b_j(z,t) = (U^b_j(z,t), O, O)$, which is periodic, parallel to the x-axis. Therefore, as in the traditional Hele-Shaw flow where the aspect ratio ϵ of the cell is considered smaller than unity, a first approximation is obtained from Eqs (1) and (2) as follows [11,12]

$$\rho_j \frac{\partial U^b_j}{\partial t} = -\frac{\partial P^b_j}{\partial x} + \mu_j \frac{\partial^2 U^b_j}{\partial z^2} + \rho_j a\omega^2 \cos(\omega t) \tag{3}$$

$$0 = -\frac{\partial P^b_j}{\partial y} - \rho_j g \tag{4}$$

Moreover, on the vertical walls, the no-slip boundaries conditions are:

$$U^b_j(z,t) = 0 \qquad \text{en } z = \pm\frac{e}{2} \tag{5}$$

The vertical end-walls at $x = 0, L$ in the horizontal hele shaw cell generate a counter-flowing layers, in order to model it, following [7], we consider the integral condition of balance of the displacement volume of both fluids below:

$$\int_{-h_1}^{0} \mathbf{V}^b_1.\mathbf{x}\,dy = -\int_{0}^{h_2} \mathbf{V}^b_2.\mathbf{x}\,dy \tag{6}$$

Equations (3)-(6) are integrated to obtain the pressure and the velocity of the basic state

$$U^b_j(z,t) = F_j(z)\cos(\omega t) + G_j(z)\sin(\omega t) \tag{7}$$

$$P^b_j = -\rho_j g y + C_j \tag{8}$$

where C_j is a constant and the functions F_j and G_j are given by:

$$F_1(z) = -\frac{h_2}{h_1}\frac{a\omega}{\nu_2}\left(\frac{\mu_1 h_2 + \mu_2 h_1}{\rho_1 h_2 + \rho_2 h_1}\right) \left[\frac{\sinh(\gamma_2\eta)\sin(\gamma_2(e-\eta)) + \sinh(\gamma_2(e-\eta))\sin(\gamma_2\eta)}{\coth(\gamma_2 e) + \cos(\gamma_2 e)}\right] \tag{9}$$

$$F_2(z) = \frac{a\omega}{\nu_2}\left(\frac{\mu_1 h_2 + \mu_2 h_1}{\rho_1 h_2 + \rho_2 h_1}\right) \left[\frac{\sinh(\gamma_2\eta)\sin(\gamma_2(e-\eta)) + \sinh(\gamma_2(e-\eta))\sin(\gamma_2\eta)}{\coth(\gamma_2 e) + \cos(\gamma_2 e)}\right] \tag{10}$$

$$G_1(z) = \frac{a\omega h_2(\rho_1 - \rho_2)}{\rho_1 h_2 + \rho_2 h_1} + \frac{h_2}{h_1}\frac{a\omega}{\nu_2}\left(\frac{\mu_1 h_2 + \mu_2 h_1}{\rho_1 h_2 + \rho_2 h_1}\right) \left[\frac{\cosh(\gamma_2\eta)\cos(\gamma_2(e-\eta)) + \cosh(\gamma_2(e-\eta))\cos(\gamma_2\eta)}{\cosh(\gamma_2 e) + \cos(\gamma_2 e)}\right] \tag{11}$$

$$G_2(z) = -\frac{a\omega h_1(\rho_1 - \rho_2)}{\rho_1 h_2 + \rho_2 h_1} - \frac{a\omega}{\nu_2}\left(\frac{\mu_1 h_2 + \mu_2 h_1}{\rho_1 h_2 + \rho_2 h_1}\right) \left[\frac{\cosh(\gamma_2\eta)\cos(\gamma_2(e-\eta)) + \cosh(\gamma_2(e-\eta))\cos(\gamma_2\eta)}{\cosh(\gamma_2 e) + \cos(\gamma_2 e)}\right] \tag{12}$$

where $\gamma_2 = (\frac{\omega}{2\nu_2})^{\frac{1}{2}}$ and $\eta = z - \frac{e}{2}$.
The averaged components of velocity can be written as:

$$\bar{U}^b_j = \overline{F}_j \cos(\omega t) + \overline{G}_j \sin(\omega t) \tag{13}$$

The functions $\overline{F}_j$ and $\overline{G}_j$ are given by:

$$\overline{F}_1 = -\frac{h_2}{h_1}\frac{a\omega}{\nu_2\Gamma_2}\left(\frac{\mu_1 h_2 + \mu_2 h_1}{\rho_1 h_2 + \rho_2 h_1}\right)\left[\frac{\sinh(\Gamma_2) - \sin(\Gamma_2)}{\coth(\Gamma_2) + \cos(\Gamma_2)}\right] \tag{14}$$

$$\overline{G}_1 = \frac{h_2}{h_1}\frac{a\omega}{\nu_2\Gamma_2}\left(\frac{\mu_1 h_2+\mu_2 h_1}{\rho_1 h_2+\rho_2 h_1}\right)\left[\frac{\sinh(\Gamma_2)+\sin(\Gamma_2)}{\cosh(\Gamma_2)+\cos(\Gamma_2)}\right]+ +a\omega\frac{h_2(\rho_1-\rho_1)}{\rho_1 h_2+\rho_2 h_1} \quad (15)$$

$$\overline{F}_2 = \frac{a\omega}{\nu_2\Gamma_2}\left(\frac{\mu_1 h_2+\mu_2 h_1}{\rho_1 h_2+\rho_2 h_1}\right)\left[\frac{\sinh(\Gamma_2)-\sin(\Gamma_2)}{\coth(\Gamma_2)+\cos(\Gamma_2)}\right] \quad (16)$$

$$\overline{G}_2 = -\frac{a\omega}{\nu_2\Gamma_2}\left(\frac{\mu_1 h_2+\mu_2 h_1}{\rho_1 h_2+\rho_2 h_1}\right)\left[\frac{\sinh(\Gamma_2)+\sin(\Gamma_2)}{\cosh(\Gamma_2)+\cos(\Gamma_2)}\right]- -a\omega\frac{h_1(\rho_1-\rho_1)}{\rho_1 h_2+\rho_2 h_1} \quad (17)$$

The parameter $\Gamma_2 = \sqrt{\frac{\omega e^2}{2\nu_2}}$ is the frequency number. In the limit case corresponding to $\Gamma_j \longrightarrow \infty$ and $\nu_j \longrightarrow 0$, our solution Eqn.13 transforms into the solution obtained by khenner et *al.*[8] in the inviscid approximation.

2.3 Perturbation equations

We assume that the base state is disturbed so that the velocity and the pressure fields in the perturbed state are written as the sum of the base flow variables and a small perturbation

$$\mathbf{V}_j = \bar{\mathbf{V}}_j^b + \mathbf{v}_j\Big(u(x,y,t), v(x,y,t)\Big)$$

$$P_j = P_j^b + p_j(x,y,t)$$

Hence the linear system of the conservation equations is written as

$$\frac{\partial u_j}{\partial x} + \frac{\partial v_j}{\partial y} = 0 \quad (18)$$

$$\rho_j\left[\frac{\partial u_j}{\partial t} + \bar{U}_j^b\frac{\partial u_j}{\partial x}\right] = -\frac{\partial p_j}{\partial x} \quad (19)$$

$$\rho_j\left[\frac{\partial v_j}{\partial t} + \bar{U}_j^b\frac{\partial v_j}{\partial x}\right] = -\frac{\partial p_j}{\partial y} \quad (20)$$

We describe the instantaneous interface as $y = 0 + \xi(x,t)$, where $\xi(x,t)$ is an infinitesimal perturbation of the horizontal interface. Hereafter, we seek the solution of the system (18)-(20) in term of normal modes as

$$[p_j, u_j, v_j] = [\tilde{p}_j(t,y), \tilde{u}_j(t,y), \tilde{v}_j(t,y)]e^{ikx}$$

$$\xi(x,t) = \tilde{\xi}(t)e^{ikx}$$

with $i^2 = -1$ and k is the wave number in the x direction. Hereafter, we consider the velocity potential $\tilde{\phi}_j$, solutions of the equation of continuity (18), defined by:

$$\tilde{\phi}_j = C_j^1(t)e^{ky} + C_j^2(t)e^{-ky} \quad (21)$$

The constants C_j^1 end C_j^2 are determined by using the non-slip boundary conditions at the horizontal walls, $v_j = 0$ at $y = -h_1$, $y = h_2$ and the kinematics condition linearized at the interface:

$$\frac{d\tilde{\xi}(t)}{dt} + ik\bar{U}_j^b\tilde{\xi}(t) = \frac{\partial\tilde{\Phi}_j}{\partial y} \quad (22)$$

To complete the set of equations we must provide the dynamic boundary condition at the interface:

$$(P_1^b + \tilde{p}_1) - (P_2^b + \tilde{p}_2) = \sigma\nabla.\mathbf{n} \quad (23)$$

The curvature of the interface is written in its linearized form as $\nabla.\mathbf{n} = k^2\tilde{\xi}(t)$. The total pressure is linearized at the interface ($y = 0$) as follows

$$(P_j^b + \tilde{p}_j) = P_j^b(0) + \frac{\partial P_j^b}{\partial y}\Big|_{y=0}\tilde{\xi} + \tilde{p}_j(y) \quad (24)$$

Finally, with the above assumptions, Eqn (23) leads to an parametric differential equation for the amplitude $\tilde{\xi}(t)$ of the interface displacement from its equilibrium position

$$\frac{d^2\tilde{\xi}(t)}{dt^2} + 2ik\frac{\left[R_1\bar{U}_1^b + R_2\bar{U}_2^b\right]}{[R_1+R_2]}\frac{d\tilde{\xi}}{dt} + \Bigg[+ \frac{(\rho_1-\rho_2)}{R_1+R_2}gk + ik\frac{R_1\frac{d\bar{U}_1^b}{dt} + R_2\frac{d\bar{U}_2^b}{dt}}{R_1+R_2} - k^2\frac{R_1(\bar{U}_1^b)^2 + R_2(\bar{U}_2^b)^2}{R_1+R_2} + \frac{\sigma k^3}{R_1+R_2}\Bigg]\tilde{\xi}(t) = 0 \quad (25)$$

with $R_1 = \rho_1\coth(kh_1)$ and $R_2 = \rho_2\coth(kh_2)$. In Eqn (25). This equation is similar to that found by Kelly [6]. It is convenient to eliminate the term which contains the first order derivative in $\tilde{\xi}(t)$Thus, we make the change of variable below

$$\tilde{\xi}(t) = \bar{\xi}(t)\exp\left[-ik\int\frac{R_1\bar{U}_1^b + R_2\bar{U}_2^b}{R_1+R_2}dt\right] \quad (26)$$

Using Eqn (26), the Eqn (25) is reduced to a nondimensional form by scaling space by $l_c = \left[\frac{\sigma}{g(\rho_1-\rho_2)}\right]^{\frac{1}{2}}$, time by $\frac{1}{\omega}$ and velocity by $a\omega$. Hence equation (25) becomes:

$$\frac{d^2\bar{\xi}(t)}{dt^2} + \frac{1}{We}\Bigg[4n^2B_v\Bigg(\frac{\left[\rho\bar{U}_1^b\coth(nH_1) + \bar{U}_2^b\coth(nH_2)\right]^2}{\left[\rho\coth(nH_1)+\coth(nH_2)\right]^2} - \frac{\rho(\bar{U}_1^b)^2\coth(nH_1) + (\bar{U}_2^b)^2\coth(nH_2)}{\rho\coth(nH_1)+\coth(nH_2)}\Bigg) + \frac{n(1-n^2)(\rho-1)}{\rho\coth(nH_1)+\coth(nH_2)}\Bigg]\bar{\xi}(t) = 0 \quad (27)$$

where $We = \frac{\omega^2 l_c}{g}$ is the Weber number based on the capillary length, $\rho = \frac{\rho_1}{\rho_2}$ is the density ratio, $n = kl_c$ and $H_j = \frac{h_j}{l_c}$ are dimensionless layer height. We also introduce the Bond number, B_v, the dimensionless parameter characterizing the vibration intensity

$$B_v = \frac{a^2\omega^2}{4}\left(\frac{\rho_1-\rho_2}{g\sigma}\right)^{\frac{1}{2}}$$

Using the change of variable $t = 2\tau$, the governing equation (27) is then approximated by the following equation

$$\frac{d^2\bar{\xi}(\tau)}{d\tau^2} + [\delta + \beta\sin(\tau) + \gamma\cos(\tau)]\,\bar{\xi}(\tau) = 0 \qquad (28)$$

which is a Mathieu equation similar to that obtained by Khenner[8] in the inviscid case without terms related to the viscosity. The coefficients of such equation are

$$\delta = \frac{n(1-n^2)(\rho-1)}{4We\alpha} + \frac{n^2B_v}{2\alpha We}\Bigg[\rho\coth(nH_1)\left[\frac{\rho}{\alpha}\coth(nH_1)-1\right](\overline{F}_1^2+\overline{G}_1^2) + \coth(nH_2)\left[\frac{\rho}{\alpha}\coth(nH_2)-1\right](\overline{F}_2^2+\overline{G}_2^2) + \frac{2\rho}{\alpha}\coth(nH_1)\coth(nH_2)(\overline{F}_1\overline{F}_2+\overline{G}_1\overline{G}_2)\Bigg]$$

$$\beta = \frac{n^2B_v}{\alpha We}\Bigg[\rho\coth(nH_1)\left[\frac{\rho}{\alpha}\coth(nH_1)-1\right]\overline{F}_1\overline{G}_1 + \coth(nH_2)\left[\frac{\rho}{\alpha}\coth(nH_2)-1\right]\overline{F}_2\overline{G}_2 + \frac{\rho}{\alpha}\coth(nH_1)\coth(nH_2)(\overline{F}_1\overline{G}_2+\overline{F}_2\overline{G}_1)\Bigg]$$

$$\gamma = \frac{n^2B_v}{2\alpha We}\Bigg[\rho\coth(nH_1)\left[\frac{\rho}{\alpha}\coth(nH_1)-1\right](\overline{F}_1^2-\overline{G}_1^2) + \coth(nH_2)\left[\frac{\rho}{\alpha}\coth(nH_2)-1\right](\overline{F}_2^2-\overline{G}_2^2) + \frac{2\rho}{\alpha}\coth(nH_1)\coth(nH_2)(\overline{F}_1\overline{F}_2-\overline{G}_1\overline{G}_2)\Bigg]$$

and

$$\alpha = \rho\coth(nH_1) + \coth(nH_2)$$

The equation (28) is solved using the Floquet theory. According to this approach, the solutions of (28) can be expressed in the form

$$(\bar{\xi})_{2\pi} = \sum_{-\infty}^{+\infty} a_n e^{in\tau} \qquad \text{Harmonic solutions} \qquad (29)$$

$$(\bar{\xi})_{4\pi} = \sum_{-\infty}^{+\infty} a_n e^{i(n+1/2)\tau} \qquad \text{Subharmonic solutions} \qquad (30)$$

The substitution of expression (29) in equation (28) leads to the Hill determinant which corresponds to the marginal stability curves $B_v(n)$. This determinant can be written formally in the form [13]

$$f(We, n, B_v, \rho, \nu, \Gamma_2, H_1, H_2) = 0 \qquad (31)$$

3 Discussion

The Mathieu equation (28) has periodic solutions which can identify regions of instability. The interface can be stabilized or destabilized depending on the frequency and the amplitude of oscillations and on other physical parameters involved in the problem. it is known that the horizontal oscillations lead to the Kelvin-Helmholtz instability which is a result of the periodic changes of the velocity, and successive parametric-resonance modes. In this work, we mainly focus on the analysis of the effect of viscosity difference between the two fluids on the onset of each mode of instability. The large friction at the vertical walls contributs to decrease the inertial effect and quite the same situation takes place in the case of a porous medium when the Darcy low is assumed for the resistance force. These observations will be clarified, on the one hand, by considering fluids of unequal viscosities and on the other hand, by comparison with a model of Lyubimov et *al*. [7] in the large frequencies and Khenner et al. [8] in the limit of inviscid fluids.

4 Conclusion

In this study, we have performed a linear stability analysis of the behavior of an interface between two viscous immiscible fluids of different densities confined in a horizontal Hele-Shaw cell under horizontal periodic oscillations. The linear problem has been reduced to a periodic Mathieu equation governing the evolution of the amplitude of the interface. This preliminary study will allow us then to determine the stable and unstable regions and modes taking into account the different physical parameters of the problem.

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