Dynamics of a forced 2 dof piece-wise linear system by consideration of the weight

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Abstract. Energy pumping phenomenon between a linear system and a non-smooth system by taking into account the weight of the system is studied. The system faces bifurcation when it reaches to its unstable border and then according to external forcing term it can follow lower stable branch or to face strongly modulated response by hysteresis jumps between its stable branches.

1 Introduction

It has been proved that a system can be passively controlled by endowing nonlinear innate of coupled cubic systems, namely nonlinear energy sink (NES) to the main one [1–20]. There has been some attempts to consider the non-smooth NES systems instead of cubic ones. From them we can mention to the work by Nucera et al. [21] for considering the vibro-impact NES, Gendelman [22] for considering the non-polynomial NES systems and Lamarque et al. [23] for considering piece-wise linear NES. In this paper we try to enlighten the system behavior in terms of the energy transfer between a linear master system and a non-smooth NES considering the weight of the whole system. The multi-scales dynamics of the system at different time scales are detected and are accompanied by numerical evidences. Organization of the paper is as it follows: The system and some analytical treatments are clarified in Sec. 2. The multi-scales dynamics of the system are detected in Sec. 3 while numerical results are collected in Sec. 4. Finally conclusions are given in Sec. 5

2 Description of the model and some analytical treatments

Let us consider the system which is preinserted in Fig.1. It consists of a main structure with the mass M which is weakly coupled to a non-smooth system with the mass m . Governing equations of the mentioned academic model under external periodic forcing term $F(t) = \Gamma \sin(\Omega t)$ are as it follows:

$$\begin{cases} M\ddot{x}_1 + k_1 x_1 + P(x_1 - x_2) + \eta(\dot{x}_1 - \dot{x}_2) + Mg = \Gamma \sin(\Omega t) \\ m\ddot{x}_2 + P(x_2 - x_1) + \eta(\dot{x}_2 - \dot{x}_1) + mg = 0 \end{cases} \quad (1)$$

P is the non-smooth potential of the NES which is an odd function and is defined as it follows:

Fig. 1. The model of the system.

$$P(z) = -\frac{\partial V(z)}{\partial z} = -F(-z) = \begin{cases} 0 & \text{if } -\delta \leq z \leq \delta \\ k_2(z - \delta) & \text{if } z \geq \delta \\ k_2(z + \delta) & \text{if } z \leq -\delta \end{cases} \quad (2)$$

The system (1) in the domain of the period T of the system ($T = t\sqrt{\frac{k_1}{M}} = t\vartheta$) can be re-written as:

$$\begin{cases} \ddot{y}_1 + y_1 + \epsilon\tilde{P}(y_1 - y_2) + \epsilon\lambda(\dot{y}_1 - \dot{y}_2) + \gamma = \epsilon f_0 \sin(\omega T) \\ \epsilon\ddot{y}_2 + \epsilon\tilde{P}(y_2 - y_1) + \epsilon\lambda(\dot{y}_2 - \dot{y}_1) + \epsilon\gamma = 0 \end{cases} \quad (3)$$

where y_1 and y_2 are shifted displacements x_1 and x_2 in domain of T and $\epsilon = \frac{m}{M}$, $\gamma = \frac{Mg}{k_1}$, $\epsilon\lambda = \frac{\eta}{\sqrt{k_1 M}}$, $\frac{1}{k_1}P(z) = \epsilon\tilde{P}(z)$, $k = \frac{1}{\epsilon} \frac{k_2}{k_1}$, $\omega = \frac{\Omega}{\vartheta}$ and $\frac{\Gamma}{k_1} = \epsilon f_0$. So, the scaled potential of the NES reads as:

$$\tilde{P}(z) = \begin{cases} 0 & \text{if } -\delta \leq z \leq \delta \\ k(z - \delta) & \text{if } z \geq \delta \\ k(z + \delta) & \text{if } z \leq -\delta \end{cases} \quad (4)$$

We are interested to study the system (3) in the vicinity of 1:1 resonance. Let us transfer the system to the following coordinates:

$$\begin{cases} v = y_1 + \epsilon y_2 \\ w = y_1 - y_2 \end{cases} \quad (5)$$

Then the system (3) takes following form:

$$\begin{cases} \ddot{v} + \frac{1}{1 + \epsilon}(v + \epsilon w) + \gamma(1 + \epsilon) = \epsilon f_0 \sin(\omega T) \\ \ddot{w} + \frac{1}{1 + \epsilon}(v + \epsilon w) + (1 + \epsilon)\tilde{P}(w) + (1 + \epsilon)\lambda\dot{w} \\ = \epsilon f_0 \sin(\omega T) \end{cases} \quad (6)$$

Let us introduce a modified form of Manevitch's complex variables [24] to the system of (6):

$$\begin{cases} B_1 + \varphi_1 e^{i\omega T} = \dot{v} + i\omega v \\ B_2 + \varphi_2 e^{i\omega T} = \dot{w} + i\omega w \end{cases} \quad (7)$$

where $B_1 = ib_1$, $B_2 = ib_2$ and $i = \sqrt{-1}$. We can present the function $\tilde{P}(w)$ in the form of Fourier series:

$$\begin{aligned} \tilde{P}(w) &= \tilde{F}\left(\frac{b_2}{\omega} - \frac{i}{2\omega}(\varphi_2 e^{i\omega T} - \varphi_2^* e^{-i\omega T})\right) \\ &= \sum_{j=-\infty}^{+\infty} f_j(b_2, \varphi_2, \varphi_2^*) e^{i\omega j T} \end{aligned} \quad (8)$$

where the $*$ represents the complex conjugate of the function under consideration. The above mentioned variables and assumptions lead us to the zero and first order averaged systems of (6) ($j = 0, 1$ in Eq. (8)):

$$\begin{cases} \frac{1}{1 + \epsilon}\left(\frac{b_1}{\omega} + \epsilon\frac{b_2}{\omega}\right) + (1 + \epsilon)\gamma = 0 \\ \frac{1}{1 + \epsilon}\left(\frac{b_1}{\omega} + \epsilon\frac{b_2}{\omega}\right) + (1 + \epsilon)f_z(b_2, \varphi_2, \varphi_2^*) = 0 \end{cases} \quad (9)$$

$$\begin{cases} \dot{\varphi}_1 = -\frac{i}{2}\epsilon f_0 + \frac{i}{2\omega(1 + \epsilon)}(\varphi_1 + \epsilon\varphi_2) - \frac{i}{2}\omega\varphi_1 \\ \dot{\varphi}_2 = -\frac{i}{2}\epsilon f_0 + \frac{i}{2\omega(1 + \epsilon)}(\varphi_1 + \epsilon\varphi_2) - \frac{\lambda(1 + \epsilon)}{2}\varphi_2 \\ -\frac{i}{2}\omega\varphi_2 - (1 + \epsilon)f_f(b_2, \varphi_2, \varphi_2^*) \end{cases} \quad (10)$$

$f_z(b_2, \varphi_2, \varphi_2^*)$ and $f_f(b_2, \varphi_2, \varphi_2^*)$ are zero and first Fourier coefficients which can be evaluated as:

$$f_z(b_2, \varphi_2, \varphi_2^*) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \tilde{P}\left(\frac{b_2}{\omega} - \frac{i}{2\omega}(\varphi_2 e^{i\omega T} - \varphi_2^* e^{-i\omega T})\right) dT \quad (11)$$

$$f_f(b_2, \varphi_2, \varphi_2^*) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \tilde{P}\left(\frac{b_2}{\omega} - \frac{i}{2\omega}(\varphi_2 e^{i\omega T} - \varphi_2^* e^{-i\omega T})\right) e^{-i\omega T} dT \quad (12)$$

It can be proved that:

$$f_f(b_2, \varphi_2, \varphi_2^*) = -\frac{i\varphi_2}{2} G_f(|\varphi_2|^2) \quad (13)$$

$$\text{If } \frac{b_2}{\omega} + \frac{|\varphi_2|}{\omega} \leq \delta$$

$$G_f(|\varphi_2|^2) = 0 \quad (14)$$

$$f_z(|\varphi_2|^2) = 0 \quad (15)$$

and if $\frac{b_2}{\omega} + \frac{|\varphi_2|}{\omega} \geq \delta$

$$G_f(|\varphi_2|^2) = \frac{k}{2\pi\omega} \left(\pi + \frac{2b_2}{|\varphi_2|^2} \sqrt{|\varphi_2|^2 - (b_2 - \delta\omega)^2} \right.$$

$$\left. - \frac{2\delta\omega}{|\varphi_2|^2} \sqrt{|\varphi_2|^2 - (b_2 - \delta\omega)^2} \right.$$

$$\left. - \frac{2b_2}{|\varphi_2|^2} \sqrt{|\varphi_2|^2 - (b_2 + \delta\omega)^2} \right) \quad (16)$$

$$\left. - \frac{2\delta\omega}{|\varphi_2|^2} \sqrt{|\varphi_2|^2 - (b_2 + \delta\omega)^2} \right.$$

$$\left. + 2 \arccos\left(\frac{b_2 + \delta\omega}{|\varphi_2|}\right) + 2 \arcsin\left(\frac{b_2 - \delta\omega}{|\varphi_2|}\right) \right)$$

$$\begin{aligned} f_z(|\varphi_2|^2) = & \frac{k}{\pi\omega} \left(b_2\pi + \right. \\ & \sqrt{(b_2 + |\varphi_2| - \delta\omega)(-b_2 + |\varphi_2| + \delta\omega)} - \\ & \left. \sqrt{(-b_2 + |\varphi_2| - \delta\omega)(b_2 + |\varphi_2| + \delta\omega)} \right) \end{aligned} \quad (17)$$

$$+ (b_2 - \delta\omega) \arcsin\left(\frac{b_2 - \delta\omega}{|\varphi_2|}\right)$$

$$+ (b_2 + \delta\omega) \arcsin\left(\frac{b_2 + \delta\omega}{|\varphi_2|}\right)$$

In order to reveal the behavior of the averaged systems of (9) and (10) an asymptotic approach [25] by introducing slow times $\tau_1, \tau_2, \dots$ with the fast time τ_0 can be implemented as follows:

$$T = \tau_0, \quad \tau_1 = \epsilon\tau_0, \quad \dots, \quad (18)$$

so,

$$\frac{d}{dT} = \frac{\partial}{\partial\tau_0} + \epsilon \frac{\partial}{\partial\tau_1} + \dots \quad (19)$$

3 Multi-scales dynamics of averaged zero and first order systems

The general form of the system (9) show that $\gamma = f_z(N_2^2)$. During the ϵ^0 order, it leads to:

$$b_1 = -\gamma \quad (20)$$

while during the ϵ^1 order we have:

$$b_2 = -\gamma(2 + \sigma) \quad (21)$$

Let us assume that $\omega = 1 + \sigma\epsilon$. The system of (10) at the ϵ^0 order yields to:

$$\frac{\partial\varphi_1}{\partial\tau_0} = 0 \Rightarrow \varphi_1 = \varphi_1(\tau_1) \quad (22)$$

$$\frac{\partial\varphi_2}{\partial\tau_0} + \frac{i(1 - G_f(|\varphi_2|^2)) + \lambda}{2} \varphi_2 = \frac{i}{2} \varphi_1 \quad (23)$$

Considering the system behavior around its fixed points ($\frac{\partial\varphi_2}{\partial\tau_0} = 0, \varphi_2 \rightarrow \Phi$) and $\varphi_1 = N_1 e^{i\delta_1}$ and $\Phi = N_2 e^{i\delta_2}$, one can reach to following invariant manifold of the system at the τ_0 time scale:

$$N_1 = N_2 \sqrt{\lambda^2 + (1 - G_f(N_2^2))^2} \quad (24)$$

It can be proved that the stability border of the above mentioned invariant manifold can be defined as [23]:

$$\lambda^2 + (1 - G_f(N_2^2))(1 - H(N_2^2)N_2 - G_f(N_2^2)) \geq 0 \quad (25)$$

where

$$H(N_2^2) = \begin{cases} 0 & \text{if } N_2 < \delta \\ \frac{2k}{\pi\omega N_2^3} \left(b_2 \left(\sqrt{N_2^2 - (b_2 + a\omega)^2} - \sqrt{N_2^2 - (b_2 - a\omega)^2} \right) + a\omega \left(\sqrt{N_2^2 - (b_2 + a\omega)^2} + \sqrt{N_2^2 - (b_2 - a\omega)^2} \right) \right) & \text{if } N_2 \geq \delta \end{cases} \quad (26)$$

An invariant manifold of a system with given parameters and its stable and unstable zones are depicted in Fig. 2.

Fig. 2. Invariant manifold of a system and its stable and unstable zones at the τ_0 time scale.

At the order of ϵ^1 the first equation of the system (10) reads as:

$$\frac{\partial \varphi_1}{\partial \tau_1} = -\frac{i}{2} f_0 + \frac{i}{2} (\Phi - \varphi_1 - \sigma \varphi_1) - \frac{i}{2} \sigma \varphi_1 \quad (27)$$

We would like to detect the behavior of the system at the infinity of τ_1 “around” the invariant manifold at the time scale τ_0 . By considering Eq. (24), the invariant manifold of the system at the τ_1 time scale can be defined as ($\frac{\partial \varphi_1}{\partial \tau_1} = 0$):

$$f_0 = N_2 \sqrt{\lambda^2 (1 + 2\sigma)^2 + \left((1 + 2\sigma) G_f(N_2^2) - 2\sigma \right)^2} \quad (28)$$

4 Numerical results

The equilibrium state of each mass of the system due to the gravity should be considered in all numerical results (x_{10} and x_{20}). It can be proved that the equilibrium state of the system is defined as:

$$\begin{cases} x_{10} = -\frac{Mg}{k_1}(1 + \epsilon) = -\gamma(1 + \epsilon) \\ x_{20} = -\frac{Mg}{k_1}(1 + \epsilon) - \delta - \frac{mg}{k_2} = -\gamma(1 + \epsilon) - \delta - \epsilon \frac{Mg}{k_2} \end{cases} \quad (29)$$

This equilibrium points should be considered in all numerical results. We can impose equilibrium states to the numerical results as its follows:

$$\begin{cases} v_{numerical} = (y_1 - x_{10}) + \epsilon(y_2 - x_{20}) \\ w_{numerical} = (y_1 - x_{10}) - (y_2 - x_{20}) \end{cases} \quad (30)$$

So, N_1 and N_2 can be defined in terms of original system of equations (N_1^{exact} and N_2^{exact}) as it follows:

$$N_1^{exact} = \sqrt{(v_{numerical})^2 + (y_1 + \epsilon y_2)^2} \quad (31)$$

$$N_2^{exact} = \sqrt{(w_{numerical})^2 + (y_1 - y_2)^2} \quad (32)$$

Let us consider following initial conditions for the system:

$$\begin{cases} y_1(0) = 1.5 + x_{10}, & \dot{x}_1(0) = 0 \\ y_2(0) = x_{20}, & \dot{x}_2(0) = 0 \end{cases} \quad (33)$$

Added numerical results to the same system of Fig. 2 for two different amplitudes of external forcing terms are depicted in Fig. 3 and Fig. 4. When the system reaches to the unstable zone, it tries to reach to other stable zone by a jump between its stable branches through a bifurcation. Then depending on the external forcing term, the system can experience strongly modulated response by hysteresis jumps between its stable branches when it reaches to its fold lines N_{21} and N_{22} (see Fig.4); this behavior leads to beating response of oscillators (see Fig. 5). The reader can refer to [23] for detailed information about bifurcations of the system between its folded singularities.

Fig. 3. Invariant manifold of the system in the presence of the gravity and corresponding numerical result: $f_0 = 0.1$

5 Conclusions

Multi-scales dynamics of a system with coupled non-smooth NES by taking into account the weight of the system is considered. A modified form of Manevitch's complex variables were considered in zero and first order averaging of the system in order to consider effect of the weight of the system. The system faces bifurcation(s) when it reaches to stability borders. Depending on the external forcing term, the bifurcation can lead the system to very low system amplitudes and/or to strongly modulated response by hysteresis jumps between its stable branches. It is worthwhile to note that other kinds of behaviors can be observed and we did not explain them in this paper and they are beyond the scope of this paper.

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Fig. 4. Invariant manifold of the system in the presence of the gravity and corresponding numerical result: $f_0 = 0.3$

Fig. 5. Beating response of the non-smooth NES: $f_0 = 0.3$

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