The Model and Quadratic Stability Problem of Buck Converter in DCM

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Abstract. Quadratic stability is an important performance for control systems. At first, the model of Buck Converter in DCM is built based on the theories of hybrid systems and switched linear systems primarily. Then quadratic stability of SLS and hybrid feedback switching rule are introduced. The problem of Buck Converter's quadratic stability is researched afterwards. In the end, the simulation analysis and verification are provided. Both experimental verification and theoretical analysis results indicate that the output of Buck Converter in DCM has an excellent performance via quadratic stability control and switching rules.

1 Introduction

Hybrid systems (HS) is defined as a unitized dynamic system interacted by discrete and continuous parts. DC-DC converters are typical HS because the operation of each mode can be regard as the continuous dynamic subsystems and the turn-on or turn-off of power switch as the discrete dynamic subsystems.

Switched linear systems (SLS), an important type of HS, have attracted considerable attention in modeling, analysis and design. Quadratic stability is one of the important problems of SLS. This problem is more complex since it depends on the switching rules as the stability of all the subsystems.

Looking from the existing literatures [1]-[5], Lyapunov theories are the dominant approaches used in study of stability. For example, the stability of DC-DC Converters in CCM (Continuous Current Mode) was analyzed in [6]. The aim of this paper is to study the quadratic stability of DC-DC converters via using Lyapunov function based on the model in DCM (Discontinuous Current Mode).

2 Modeling of buck converter in DCM

2.1 Buck converter in DCM

The topology structure of Buck converter is shown in Fig.1. Assume all of the components are perfect. Three work modes of it in DCM are shown in Fig. 2.

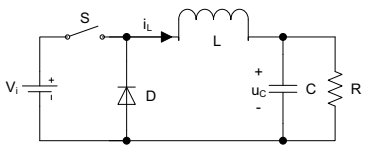
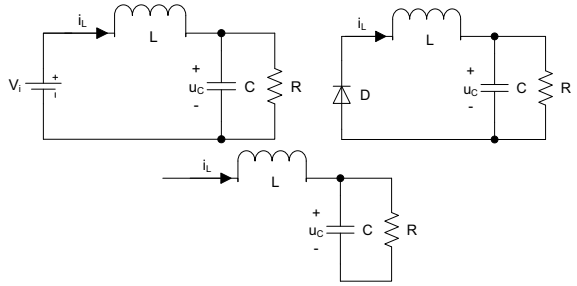


Figure 1. Buck converter.



(a) Switch tube is on (b) Switch tube is off and diode is on (c) Switch tube and diode are all off

Figure 2. Buck Converter in DCM

2.2 SLS Model of Buck Converter in DCM

Supposing all of the components are ideal and the state vector is $x(t) = [\ i_L \quad u_C\ ]^T$,the output vector is $y(t) = u_C$, consequently, the state equations of Buck Converter in SLS model are[7]

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_j \boldsymbol{x}(t) + \boldsymbol{B}_j \\ \boldsymbol{y}(t) = \boldsymbol{C}_j \boldsymbol{x}(t) + \boldsymbol{D}_j \end{cases} \tag{1}$$

The parameters matrixes of Buck Converter in DCM are expressed as follows according to Fig. 2.

$$\text{(a)}\ A_1 = \begin{bmatrix} 0 & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix},\ B_1 = \begin{bmatrix} \dfrac{V_{in}}{L} \\ 0 \end{bmatrix},$$

$$C_1 = [0 \quad 1\ ],\ D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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(b) $A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$C_2 = [0 \quad 1]$, $D_2 == \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$C_3 = [0 \ 1]$, $D_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So it can be seen that Buck Converter in DCM has three switched modes, (A_1 , B_1 , C_1 ,D_1), (A_2 , B_2 , C_2 ,D_2) and (A_3 , B_3 , C_3 ,D_3),which belong to three subsystems respectively, Σ_1, Σ_2 and Σ_3.

3 Quadratic stability of SLS

3.1 Propaedeutics

Given a stable equilibrium point $\bar{x}$. Since any other equilibrium point can be shifted to the origin via a change of variable $\tilde{x} = x - \bar{x}$, we can assume the switched equilibrium is the origin $\bar{x}=0$ without loss of generality.

Definition: if and only if there exist a matrix $P = P^{\mathrm{T}} > 0$ and a constant $\varepsilon > 0$ such that for the quadratic function $V(x) = x^T Px$, we have $\dot{V}(x(t)) \leqslant \varepsilon x^T x$ along all system trajectories, the switched equilibrium $\bar{x} = 0$ can be said quadratically stable[8].

(a) when subsystem Σ_i is active,

$$\dot{V}(x(t)) = \dot{V}_i(x(t)) = x^T(A_i^T P + PA_i)x + 2b_i^T Px,$$

(b) at the switching point between Σ_i and Σ_j,

$$\dot{V}(x(t)) = \sup_{\gamma \in [0,1]} \{\gamma \dot{V}_i(x(t)) + (1 \quad \gamma)\dot{V}_j(x(t))\}.$$

If we denote $A_{eq} = \sum_{i=1}^m \alpha_i A_i$ and $b_{eq} = \sum_{i=1}^m \alpha_i b_i$,where $0<\alpha_i<1$ and $\sum_{i=1}^m \alpha_i = 1$, a convex combination of the subsystems is defined as

$$\Sigma_{eq} : \dot{x} = A_{eq}x + b_{eq}. \tag{2}$$

3.2 Quadratic stability

Theorem 1: As for system (l), the point $\bar{x} = 0$ is a quadratic stable switched equilibrium if there exist $\alpha_i \in (0,1)$, i =1,...,m

$$\text{such that } \sum_{i=1}^m \alpha_i = 1, \tag{3}$$

$$A_{eq} = \sum_{i=1}^m \alpha_i A_i \text{ is Hurwitz,} \tag{4}$$

$$b_{eq} = \sum_{i=1}^m \alpha_i b_i = 0 \tag{5}$$

Since the convex combination A_{eq} in Eq. 4 is stable, there exist two positive definite symmetric matrices P and Q such that

$$A_{eq}^T P + PA_{eq} = Q. \tag{6}$$

According to Eq. 4, Eq. 6 can be rewritten as

$$\sum_{i=1}^m \alpha_i x^T(A_i P + PA_i)x = x^T Qx \tag{7}$$

From Eq. 5, we can also get the null term

$$2(\alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_m b_m)^T Px = 0 \tag{8}$$

Now, a new equation is obtained as follows

$$\begin{aligned} &\alpha_1(x^T(A_1^T P + PA_1)x + 2b_1^T Px) + \alpha_2(x^T(A_2^T P + PA_2)x + 2b_2^T Px) \\ &+ ... + \alpha_m(x^T(A_m^T P + PA_m)x + 2b_m^T Px) = x^T Qx \leqslant \varepsilon x^T x \end{aligned} \tag{9}$$

where $0<\varepsilon \leq \lambda_{\min}$ and $\lambda_{\min}$ is the smallest positive real eigenvalue of Q. Then Eq. 9 is equivalent to Eq. 10 as follows.

$$\begin{aligned} &\alpha_1(x^T(A_1^T P + PA_1)x + 2b_1^T Px + \varepsilon x^T x) \\ &+ \alpha_2(x^T(A_2^T P + PA_2)x + 2b_2^T Px + \varepsilon x^T x) \\ &+ ... + \alpha_m(x^T(A_m^T P + PA_m)x + 2b_m^T Px + \varepsilon x^T x) \leqslant 0 \end{aligned} \tag{10}$$

4 Hybrid feedback switching rule

In this rule, a lower bound on the decay rate $\dot{V}_{\min}(x(t)) = -\varepsilon x^T x$ with $\varepsilon > 0$ is presented primarily, then the active subsystem is switched off only when it no more satisfies the required constraint.

The procedures of hybrid feedback switching rule is listed as follows.

(*initialization*) at time t = 0 activate the subsystem Σ_{i_0} with $i_0 = \arg\min\{(\dot{V}_i(x_0))\}$.

(*switching off rule*) if subsystem Σ_i is active and $x^T(A_i^T P + PA_i)x + 2b_i^T Px > -\varepsilon x^T x$,the system will switch to subsystem Σ_j with $j = \arg\min\{(\dot{V}_j(x))\}$.

(*equilibrium neighborhood rule*) if $\|x\| \leq \rho_{off}$ switching will be stopped until $\|x\| \geq \rho_{on}$ ($\rho_{off} < \rho_{on}$).

5 Quadratic stability of buck converter

5.1 Coordinate Transformation of Buck Converter

Assume the anticipant output $\bar{x} = [\bar{x}_1 \quad \bar{x}_2]^T$ as the switched equilibrium, we use $\tilde{x} = x - \bar{x}$ for conversion of coordinates to transfer switched equilibrium to original point. The state equations after coordinate transformation is given by Eq. 11 and Table 1.

Table 1. Parameters matrixes of buck converter after coordinate transformation

Σ_1 :w=1, v=1 (Switch tube is on)	Σ_2 :w=1, v=0 (Switch tube is off and diode is on)	Σ_3 :w=0, v=0 (Switch tube and diode are all off)
$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$	$A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$	$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}$
$B_1 = \begin{bmatrix} \frac{V_{in}}{L} - \frac{1}{L}\bar{x}_2 \\ \frac{1}{C}\bar{x}_1 - \frac{1}{RC}\bar{x}_2 \end{bmatrix}$	$B_2 = \begin{bmatrix} -\frac{\bar{x}_2}{L} \\ \frac{1}{C}\bar{x}_1 - \frac{1}{RC}\bar{x}_2 \end{bmatrix}$	$B_3 = \begin{bmatrix} 0 \\ -\frac{1}{RC}\bar{x}_2 \end{bmatrix}$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L}w \\ \frac{1}{C}w & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{L}v - \frac{1}{L}w\bar{x}_2 \\ \frac{1}{C}w\bar{x}_1 - \frac{1}{RC}\bar{x}_2 \end{bmatrix} \tag{11}$$

5.2 Switching control of buck converter

Buck Converter in DCM includes three subsystem, consequently there exist three coefficients α_1, α_2 and α_3 in the convex combination of Eq.3 .According to Eq.3, we have α_3=1-(α_1+α_2). Then the convex combination can be written as

$$A_{eq} = \begin{bmatrix} 0 & -\frac{\alpha_1 + \alpha_2}{L} \\ \frac{\alpha_1 + \alpha_2}{C} & -\frac{1}{RC} \end{bmatrix},$$

$$b_{eq} = \begin{bmatrix} \frac{\alpha_1 V_{in} - (\alpha_1 + \alpha_2)\bar{x}_2}{L} \\ \frac{(\alpha_1 + \alpha_2)R\bar{x}_1 - \bar{x}_2}{RC} \end{bmatrix}.$$

Assume $P = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}$, the Lyapunov function is

$$V(\tilde{x}) = \tilde{x}^T P \tilde{x} = \frac{1}{2}L\tilde{x}_1^2 + \frac{1}{2}C\tilde{x}_2^2 = \frac{1}{2}Li_L^2 + \frac{1}{2}Cu_C^2.$$

Obviously $V(\tilde{x})$ is an energy function. Then we get

$$A_{eq}^T P + PA_{eq} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2}{R} \end{bmatrix} = -Q.$$

Where Q is semi-definite positive. Q can be chosen as a semi-definite matrix because $\dot{V}(\tilde{x})$ is not identically vanishing along all system trajectories for DC-DC converters. Let $\lambda_{\min}$ =2/R is the smallest positive real eigenvalue of Q and $0<\varepsilon \leq \lambda_{\min}$. So the regions of subsystems can be divided by hybrid feedback rule.

5.3 Simulation of buck converter

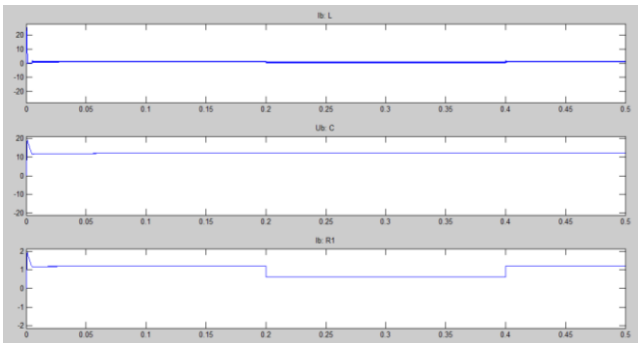

Figure 3. Simulation results of buck converter on hybrid feedback switching control rule.

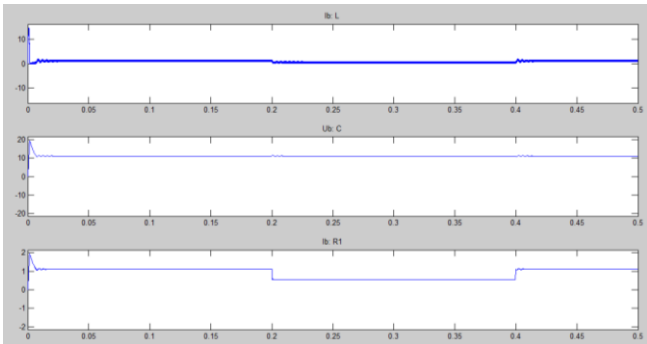

Figure 4. Simulation results of buck converter in open-loop control.

In this section, simulation results of Buck converter based on hybrid feedback switching rule are presented in Fig. 3(V_{in}=24V, f=20KHz, R=10Ω, L=400μH, C=750μF). Fig. 4 demonstrates the simulation results when Buck Converter is open-loop as a compared object.

The load resistor *R* has two salutations during the process of simulation, one is from 10Ω to 20Ω when t=0.2s, the other is from 20Ω to 10Ω when t=0.4s. According to Fig.4,we can see the output voltage reach its steady state with some overshoots and it spends more time to stabilize compared with the converter in Fig.3 when the load resistor changed. So the hybrid feedback switching rule appropriates good performance of transient and steady state dynamic responses.

6 Conclusion

In this paper, the model of Buck Converter in DCM, which is based on the concept and theory of SLS, is built first and foremost. Then the quadratic stability and switching rule of it are studied. Afterwards, simulation results are presented. This approach can also be extended to other converters and port controlled Hamilton linear switched system. The research results are beneficial to the development of nonlinear control strategy and the practical applications of power electronic systems.

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